

Plurisubharmonic functions on the octonionic plane and $Spin(9)$ -invariant valuations on convex sets.

Semyon Alesker *

Department of Mathematics, Tel Aviv University, Ramat Aviv

69978 Tel Aviv, Israel

e-mail: semyon@post.tau.ac.il

Abstract

A new class of plurisubharmonic functions on the octonionic plane $\mathbb{O}^2 \simeq \mathbb{R}^{16}$ is introduced. An octonionic version of theorems of A.D. Aleksandrov [3] and Chern-Levine-Nirenberg [24], and Blocki [21] are proved. These results are used to construct new examples of continuous translation invariant valuations on convex subsets of $\mathbb{O}^2 \simeq \mathbb{R}^{16}$. In particular a new example of $Spin(9)$ -invariant valuation on \mathbb{R}^{16} is given.

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0 Introduction

0.1 An overview

Let \mathbb{O} denote the division algebra of octonions (=Cayley numbers). The goal of this article is to introduce and study a new class of plurisubharmonic functions on the octonionic plane $\mathbb{O}^2 \simeq \mathbb{R}^{16}$, and to give some applications to the theory of valuations on convex sets. In particular octonionic versions of theorems of A.D. Aleksandrov [3] and Chern-Levine-Nirenberg [24], and Błocki [21] are proved. These results are used to construct new examples of continuous translation invariant valuations on convex subsets of $\mathbb{O}^2 \simeq \mathbb{R}^{16}$. In particular a new example of a $Spin(9)$ -invariant valuation on \mathbb{R}^{16} is given (notice that $Spin(9)$ is one of the three exceptional examples of compact connected groups acting transitively on a sphere - see discussion below in the introduction).

The theories of convex functions on \mathbb{R}^n and plurisubharmonic functions on complex manifolds are classical and well studied: see e.g. the book [34] for the introduction to these subjects; the book by Lelong [37] is a classical introduction to the theory of plurisubharmonic functions of complex variables.

More recently a class of plurisubharmonic functions of quaternionic variables on the flat quaternionic space \mathbb{H}^n has been introduced independently and at the same time by the author and G. Henkin. It was investigated further and applied by the author [6], [7],[11] and G. Henkin [33]. Then part of this theory has been generalized to more general context of (not necessarily flat) hypercomplex manifolds by M. Verbitsky and the author [16]. We refer also to [13] for a survey of these results. Very recently, other classes of plurisubharmonic functions have been introduced in the context of calibrated geometries [32].

Let us describe our main results in greater details. The algebra \mathbb{O} of octonions is a non-associative non-commutative division algebra over the reals \mathbb{R} of dimension 8. \mathbb{O} has a basis over \mathbb{R} : e_0, e_1, \dots, e_7 where $e_0 = 1$ is the identity element, and for $i > 0$ the e_i 's are anti-commuting elements satisfying $e_i^2 = -1$. Any octonion $q \in \mathbb{O}$ can be uniquely written in the form

$$q = \sum_{i=0}^7 x_i e_i \text{ with } x_i \in \mathbb{R}.$$

One introduces an *octonionic conjugation* by $q \mapsto \bar{q} := x_0 - \sum_{i=1}^7 x_i e_i$. This conjugation is an anti-involution on \mathbb{O} . These and some other elementary properties of \mathbb{O} are reviewed in more details in Section 1. For further properties we refer to the survey article [18] and Chapters 6,14 of the book [31].

Let $F: \mathbb{O} \rightarrow \mathbb{O}$ be a smooth function. We introduce two *Dirac operators* as follows

$$\frac{\partial}{\partial \bar{q}} F := \sum_{i=0}^7 e_i \frac{\partial F}{\partial x_i}, \quad (1)$$

$$\frac{\partial}{\partial q} F := \overline{\frac{\partial F}{\partial \bar{q}}} = \sum_{i=0}^7 \frac{\partial F}{\partial x_i} \bar{e}_i. \quad (2)$$

0.1.1 Remark. These operators have been introduced in analogy to the complex and quaternionic cases where they are well known (see [6] and references therein).

Similarly if $F: \mathbb{O}^m \rightarrow \mathbb{O}$ is a smooth function of m octonionic variables then one can define operators $\frac{\partial}{\partial q_i}, \frac{\partial}{\partial \bar{q}_i}$ for $i = 1, \dots, m$. It is easy to see that if the function F is real valued then for any $i, j = 1, \dots, m$

$$\frac{\partial}{\partial q_i} \left(\frac{\partial}{\partial \bar{q}_j} F \right) = \frac{\partial}{\partial \bar{q}_j} \left(\frac{\partial}{\partial q_i} F \right).$$

This expression will be denoted either by $\frac{\partial^2 F}{\partial q_i \partial \bar{q}_j}$ or by $\frac{\partial^2 F}{\partial \bar{q}_j \partial q_i}$. The matrix with octonionic entries $\left(\frac{\partial^2 F}{\partial q_i \partial \bar{q}_j} \right)_{i,j=1}^m$ will be called the *octonionic Hessian* of a real valued function F .

In fact the octonionic Hessian of a real valued function is an octonionic hermitian matrix. A matrix $A = (a_{ij})$ is called octonionic hermitian if

$$a_{ij} = \overline{a_{ji}} \text{ for any } i, j.$$

Let us discuss now the case of $m = 2$ octonionic variables (the case $m = 1$ is trivial, the case $m \geq 3$ seems to be quite different from the case $m = 2$ and it has not been studied). Let $\Omega \subset \mathbb{O}^2$ be an open subset.

0.1.2 Definition. A function $f: \Omega \rightarrow \mathbb{R} \cup \{-\infty\}$ is called *octonionic plurisubharmonic* if f is upper semi-continuous and its restrictions to any affine octonionic line is subharmonic.

Affine octonionic lines are discussed in Section 2 below; relevant notions used in Definition 0.1.2 are recalled in Section 3.1.

It is shown in Proposition 3.1.8 that a twice continuously differentiable function f is octonionic plurisubharmonic if and only if the octonionic Hessian $\left(\frac{\partial^2 f}{\partial q_i \partial \bar{q}_j} \right)_{i,j=1}^2$ is non-negative definite octonionic hermitian matrix pointwise in the sense of Definition 1.2.3 below. Notice that any convex function is octonionic plurisubharmonic. Any octonionic plurisubharmonic function is subharmonic (Proposition 3.1.6). The sum and the maximum of finitely many octonionic plurisubharmonic functions are octonionic plurisubharmonic (Proposition 3.1.11). The class of octonionic plurisubharmonic functions is invariant under translations and linear transformations from the group $SL_2(\mathbb{O}) \simeq Spin(9, 1)$ (which is discussed in Section 1.4). We denote by $P(\Omega)$ the class of all octonionic plurisubharmonic functions in Ω .

On the class of octonionic hermitian (2×2) -matrices there exists a determinant function \det with various nice properties completely analogous to properties of the usual determinant

of real symmetric and complex hermitian matrices (and also of the Moore determinant of quaternionic hermitian matrices). We refer to Section 1.2 for the definition and the main properties of it. This determinant plays an important role in our constructions. Note also that for octonionic hermitian matrices of size at least 4 no nice notion of determinant is known, while for matrices of size 3 it does exist (see e.g. Section 3.4 in [18]).

The first main result can be stated as follows (see Proposition 3.2.5 and Theorem 3.2.7).

0.1.3 Theorem. *For any $f \in P(\Omega) \cap C(\Omega)$ there exists a non-negative measure in Ω denoted by $\det \left(\frac{\partial^2 f}{\partial q_i \partial \bar{q}_j} \right)$ which is uniquely characterized by the following two properties:*

- (i) *this measure has the obvious meaning if $f \in C^2(\Omega)$;*
- (ii) *if a sequence $\{f_n\} \subset P(\Omega) \cap C(\Omega)$ converges uniformly on compact subsets to the function f then*

$$\det \left(\frac{\partial^2 f_n}{\partial q_i \partial \bar{q}_j} \right) \rightarrow \det \left(\frac{\partial^2 f}{\partial q_i \partial \bar{q}_j} \right)$$

weakly in sense of measures.

0.1.4 Remark. A real version of this result for the usual Hessian of convex functions was proved by A.D. Aleksandrov [3]. A complex version for the complex Hessian of complex plurisubharmonic functions was proved by Chern-Levine-Nirenberg [24]. A quaternionic version for quaternionic Hessian of quaternionic plurisubharmonic functions was proved by the author [6] on the flat space \mathbb{H}^n and by M. Verbitsky and the author [16] on more general hypercomplex manifolds.

The second main result is the following octonionic version of a result by Błocki [21] for complex plurisubharmonic functions (a quaternionic version was proved by the author in [11]).

0.1.5 Theorem. *For any $u, v \in P(\Omega) \cap C(\Omega)$ such that $\min\{u, v\} \in P(\Omega)$ one has*

$$\det(\partial^2(\min\{u, v\})) = \det(\partial^2 u) + \det(\partial^2 v) - \det(\partial^2(\max\{u, v\})).$$

Actually this result is an easy consequence of a more precise Theorem 3.3.1 (which is due to Błocki [21] in the complex case).

We refer to Section 3 for other results on octonionic plurisubharmonic functions. Now we are going to describe applications of the above results to the theory of valuations on convex sets. First let us remind basic notions of this theory referring for further information to the surveys by McMullen [40] and McMullen and Schneider [41]. Let V be a finite dimensional real vector space. Let $\mathcal{K}(V)$ denote the class of all non-empty convex compact subsets of V .

0.1.6 Definition. (i) A function $\phi : \mathcal{K}(V) \rightarrow \mathbb{C}$ is called a *valuation* if for any $K_1, K_2 \in \mathcal{K}(V)$ such that their union is also convex one has

$$\phi(K_1 \cup K_2) = \phi(K_1) + \phi(K_2) - \phi(K_1 \cap K_2).$$

(ii) A valuation ϕ is called *continuous* if it is continuous with respect the Hausdorff metric on $\mathcal{K}(V)$.

Recall that the Hausdorff metric d_H on $\mathcal{K}(V)$ depends on a choice of a Euclidean metric on V and it is defined as follows: $d_H(A, B) := \inf\{\varepsilon > 0 | A \subset (B)_\varepsilon \text{ and } B \subset (A)_\varepsilon\}$ where $(U)_\varepsilon$ denotes the ε -neighborhood of a set U . Equipped with the Hausdorff metric, $\mathcal{K}(V)$ becomes a locally compact space, and the induced topology on $\mathcal{K}(V)$ is independent of a choice of the Euclidean metric on V . Let us denote by $Val(V)$ the space of translation invariant continuous valuations on V .

0.1.7 Example. (1) A Lebesgue measure vol on V belongs to $Val(V)$.

(2) The Euler characteristic χ belongs to $Val(V)$. Recall that $\chi(K) = 1$ for any $K \in \mathcal{K}(V)$.

(3) Denote $m := \dim V$. Fix $k = 1, \dots, m$. Fix $A_1, \dots, A_{m-k} \in \mathcal{K}(V)$. Then the mixed volume

$$K \mapsto V(K[k], A_1, \dots, A_{m-k})$$

belongs to $Val(V)$ (here $K[k]$ means that a set K is taken k times). For the notion of mixed volume and its properties see e.g. the book [45].

It was conjectured by P. McMullen [39] and proved by the author [5] that the linear combinations of the mixed volumes as in Example 0.1.7 (3) above are dense in $Val(V)$ in the topology of uniform convergence on compact subsets of $\mathcal{K}(V)$. Nevertheless there are other than mixed volumes non-trivial constructions of translation invariant continuous valuations. One of such constructions will be described in Theorem 0.1.8 below, which provides in particular a new example of a continuous $Spin(9)$ -invariant valuation on \mathbb{R}^{16} .

To explain why such examples are interesting let us digress to a more general context of valuations invariant under a group. Assume that V is a Euclidean space. Let G be a compact subgroup of the orthogonal group. Let us denote by $Val^G(V)$ the space of G -invariant continuous translation invariant valuations. It is known [4] that the space $Val^G(V)$ is finite dimensional if and only if G acts transitively on the unit sphere in V . In such a case one can try to classify explicitly this space.

In topology there exists a complete classification of compact connected Lie groups acting transitively on spheres [42], [22], [23]. It is shown that there exist

- 6 infinite series: $SO(n), U(n), SU(n), Sp(n), Sp(n) \cdot Sp(1), Sp(n) \cdot U(1)$;
- 3 exceptions: $G_2, Spin(7), Spin(9)$.

If G is either the full orthogonal or special orthogonal group the corresponding classification of G -invariant valuations is well known and this is the famous result by Hadwiger [30]. The classification for the unitary group $U(n)$ acting on $\mathbb{C}^n \simeq \mathbb{R}^{2n}$ was obtained by the author [8] (see Fu's article [27] for further information on the algebra structure of $Val^{U(n)}(\mathbb{C}^n)$). The case of $G = SU(2)$ has been classified by the author [10] (see Bernig's article [19] for the algebra structure of $Val^{SU(2)}(\mathbb{C}^2)$).

Except of these groups $O(n), SO(n), U(n)$, and $SU(2)$ no classification of valuations has been obtained so far. To obtain such a classification is an interesting question which does not seem to have an easy solution. In [11] some non-trivial examples of valuations invariant under the quaternionic groups have been constructed. The motivation of this article is to say something on the exceptional group $Spin(9)$ which acts transitively on the sphere $S^{15} \subset \mathbb{R}^{16}$. This group is discussed in Section 1.3, see in particular Remark 1.4.2. It will be convenient to identify \mathbb{R}^{16} with the octonionic plane \mathbb{O}^2 .

The next main result of this article is a new construction of continuous valuations on $\mathbb{O}^2 \simeq \mathbb{R}^{16}$ based on octonionic plurisubharmonic functions. For a convex set $K \in \mathcal{K}(\mathbb{O}^2)$ we denote its supporting functional by $h_K: \mathbb{O}^2 \rightarrow \mathbb{R}$ (the definition is recalled in Section 4.2).

0.1.8 Theorem. *Fix a continuous compactly supported function ψ on \mathbb{O}^2 . Then*

$$K \mapsto \int_{\mathbb{O}^2} \det \left(\frac{\partial^2 h_K}{\partial q_i \partial \bar{q}_j} \right) \cdot \psi dq$$

is a translation invariant continuous valuation on $\mathcal{K}(\mathbb{O}^2)$.

The continuity is a consequence of Theorem 0.1.3, and the valuation property is a consequence of Theorem 0.1.5. As an immediate corollary we get that

$$P_{\mathbb{O}}(K) := \int_D \det \left(\frac{\partial^2 h_K}{\partial q_i \partial \bar{q}_j} \right) dq,$$

where D is the unit centered ball in \mathbb{O}^2 , is a continuous translation invariant $Spin(9)$ -invariant valuation. We call $P_{\mathbb{O}}$ the *octonionic pseudo-volume*. Other examples of $Spin(9)$ -invariant valuations of different nature coming from the convex and integral geometry are described in Section 4.1.

It should be noted that a complex (and in fact the original) version of the pseudo-volume using the complex Hessian was first considered in the context of convexity (though not of valuations) by Kazarnovskii[35], [36]. The quaternionic version of the pseudo-volume using the quaternionic Hessian was constructed by the author in [11]. As a side remark notice that the real version of the pseudo-volume is proportional to the usual volume.

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0.2 Organization on the article

The article is organized as follows. In Section 1 we collect various definitions and facts related to octonions including some linear algebra over octonions. Probably no result of this section is new (with the only possible exception of Proposition 1.5.1). A reader familiar with octonions can skip this section and consult it only whenever necessary.

In Section 2 we prove that the Radon transform over the set of affine octonionic lines in \mathbb{O}^2 is injective. We prove it by constructing an explicit inversion formula. Most probably this result is not new. We need it for some technical reasons (proof of Lemma 3.2.1) and present a proof due to the lack of a reference.

Sections 3 and 4 are the main ones. In Section 3 we introduce the class of plurisubharmonic functions on the octonionic plane \mathbb{O}^2 and establish our main results on this class.

In Section 4 we discuss in detail applications of the above technique to valuations on convex sets in $\mathbb{O}^2 \simeq \mathbb{R}^{16}$ and describe some other $Spin(9)$ -invariant valuations..

0.3 Notation list.

The following notation will be used often throughout the article.

- \mathbb{H}, \mathbb{O} - algebras of quaternions and octonions respectively;
- $\left(\frac{\partial^2 u}{\partial q_i \partial \bar{q}_j}\right)$ or $(\partial^2 u)$ - the octonionic Hessian of a real valued function u .
- $C(\Omega)$ the space of continuous functions in a domain Ω .
- $C^\infty(\Omega)$ (resp. $C_0^\infty(\Omega)$) - the space of complex valued infinitely smooth functions (resp. with compact support) in Ω .
- $C^{-\infty}(\Omega)$ - the space of complex valued generalized functions in Ω (by definition, this space is the topological dual to the space of infinitely smooth densities with compact support in Ω).
- $L_{loc}^1(\Omega)$ - the space of locally integrable functions in Ω .
- $P(\Omega)$ - the space of octonionic plurisubharmonic functions in $\Omega \subset \mathbb{O}^2$.
- $\mathbb{O}\mathbb{P}^1$ - the octonionic projective line (see Section 1.3).
- $\mathcal{K}(V)$ - the family of non-empty convex compact subsets of a vector space V .
- $Val(V)$ - the space of translation invariant continuous valuations on convex compact subsets of V .
- $Val^G(V)$ - the space of translation invariant continuous valuations on V which are invariant under a group G .

1 Basic properties of the octonions.

In this section we collect various facts on octonions. Probably no result in this section is new with the only possible exception of Proposition 1.5.1. Whenever possible we give references. Otherwise we present a proof. The reader is advised to consult the survey article [18] and Chapters 6,14 of the book [31].

1.1 Some octonionic algebra.

The octonions \mathbb{O} form an 8-dimensional algebra over the reals \mathbb{R} which is neither associative nor commutative. The product can be described as follows. \mathbb{O} has a basis e_0, e_1, \dots, e_7 over \mathbb{R} where $e_0 = 1$ is the unit, and the product can be given by the multiplication table:

	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	-1	e_4	e_7	$-e_2$	e_6	$-e_5$	$-e_3$
e_2	$-e_4$	-1	e_5	e_1	$-e_3$	e_7	$-e_6$
e_3	$-e_7$	$-e_5$	-1	e_6	e_2	$-e_4$	e_1
e_4	e_2	$-e_1$	$-e_6$	-1	e_7	e_3	$-e_5$
e_5	$-e_6$	e_3	$-e_2$	$-e_7$	-1	e_1	e_4
e_6	e_5	$-e_7$	e_4	$-e_3$	$-e_1$	-1	e_2
e_7	e_3	e_6	$-e_1$	e_5	$-e_4$	$-e_2$	-1

There is also another easier way to remember the product using so called *Fano plane* (see Figure 1 below). In Figure 1 each pair of distinct points lies in a unique line (the circle is also considered to be a line). Each line contains exactly three points, and these points are cyclically oriented. If e_i, e_j, e_k are cyclically oriented in this way then

$$e_i e_j = -e_j e_i = e_k.$$

We have to add two more rules:

- $e_0 = 1$ is the identity element;
- $e_i^2 = -1$ for $i > 0$.

All these rules define uniquely the algebra structure of \mathbb{O} . The center of \mathbb{O} is equal to \mathbb{R} .

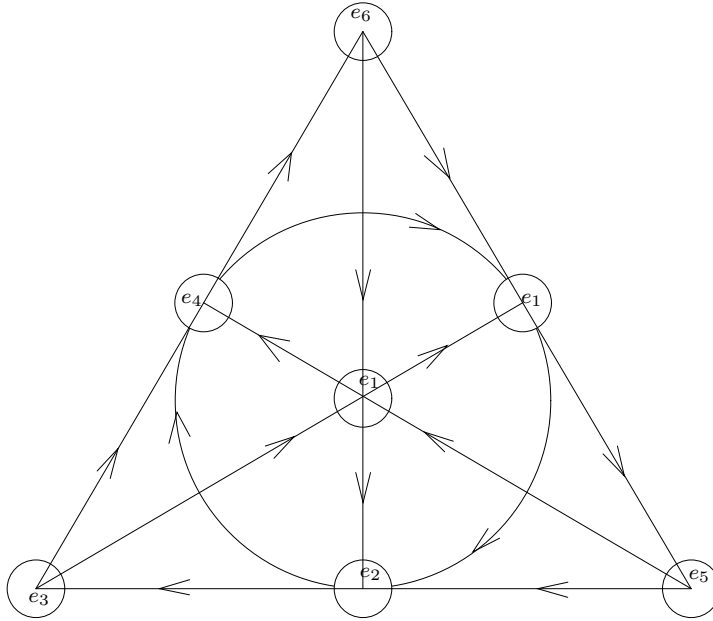


Figure 1

Every octonion $q \in \mathbb{O}$ can be written uniquely in the form

$$q = \sum_{i=0}^7 x_i e_i$$

where $x_i \in \mathbb{R}$. The summand $x_0 e_0 = x_0$ is called the real part of q and is denoted by $Re(q)$.

One defines the octonionic conjugate of q by

$$\bar{q} := x_0 - \sum_{i=1}^7 x_i e_i.$$

It is well known that the conjugation is an anti-involution of \mathbb{O} :

$$\bar{\bar{q}} = q, \overline{a+b} = \bar{a} + \bar{b}, \overline{ab} = \bar{b}\bar{a}.$$

Let us define a norm on \mathbb{O} by $|q| := \sqrt{q\bar{q}}$. Then $|\cdot|$ is a multiplicative norm on \mathbb{O} : $|ab| = |a||b|$. The square of the norm $|\cdot|^2$ is a positive definite quadratic form. Its polarization is a positive definite scalar product $\langle \cdot, \cdot \rangle$ on \mathbb{O} which is given explicitly by

$$\langle x, y \rangle = \operatorname{Re}(x\bar{y}).$$

Furthermore \mathbb{O} is a division algebra: any $q \neq 0$ has a unique inverse q^{-1} such that $qq^{-1} = q^{-1}q = 1$. In fact

$$q^{-1} = |q|^{-2}\bar{q}.$$

We denote by \mathbb{H} the usual quaternions. It is associative division algebra. We will fix once and for all an imbedding of algebras $\mathbb{H} \subset \mathbb{O}$. Let us denote by $i, j \in \mathbb{H}$ the usual quaternionic units, and $k = ij$. Then i, j, k are pairwise orthogonal with respect to the scalar product $\langle \cdot, \cdot \rangle$. Let us fix once and for all an octonionic unit $l \in \mathbb{O}$, $l^2 = -1$ which is orthogonal to i, j, k . Then l anti-commutes with i, j, k . Every element $q \in \mathbb{O}$ can be written uniquely in the form

$$q = x + yl \text{ with } x, y \in \mathbb{H}.$$

Then we can multiply two such octonions using the formula

$$(x + yl)(w + zl) = (xw - \bar{z}y) + (zx + y\bar{w})l \quad (3)$$

where $x, y, w, z \in \mathbb{H}$.

We have the following weak forms of the associativity in octonions.

1.1.1 Lemma. *Let $a, b, c \in \mathbb{O}$. Then*

- (i) $\operatorname{Re}((ab)c) = \operatorname{Re}(a(bc))$ (this real number will be denoted by $\operatorname{Re}(abc)$).
- (ii) $a(bc) + \bar{b}(\bar{a}c) = (ab + \bar{b}\bar{a})c$.
- (iii) $(ca)b + (\bar{c}\bar{b})\bar{a} = c(ab + \bar{b}\bar{a})$.
- (iv) Any subalgebra of \mathbb{O} generated by any two elements and their conjugates is associative. It is always isomorphic either to \mathbb{R} , \mathbb{C} , or \mathbb{H} .
- (v) $\operatorname{Re}((\bar{a}b)(ca)) = |a|^2 \operatorname{Re}(bc)$.

Proof. For (i) see Corollary 15.12(i) in [1]. For (iv) see e.g. Chapter 15 in [1], particularly Lemma 15.6 which is essentially equivalent to statement (iv) of our lemma.

Observe that (iii) is equivalent to (ii) by taking the conjugation. Thus let us prove (ii). Let us define the 3-linear map $[\cdot, \cdot, \cdot]: \mathbb{O} \times \mathbb{O} \times \mathbb{O} \rightarrow \mathbb{O}$ called associator which is defined by $[x, y, z] := (xy)z - x(yz)$. Then the statement (ii) is equivalent to

$$[a, b, c] = -[\bar{b}, \bar{a}, c]. \quad (4)$$

By Theorem 15.11(ii) of [1] the associator $[x, y, z]$ changes sign when one conjugates any variables. By Theorem 15.11(iii) of [1] the associator $[x, y, z]$ is an alternating function of three variables. These two properties imply the identity (4).

Let us prove part (v). It is clear that both sides of the equality we have to prove, are linear in b, c . By the part (iv) they are equal to each other (even without taking the real part) in the following two cases: (i) at least one of b and c is real; (ii) b and c are proportional to each other (with a real coefficient). Hence we may assume that $Re(b) = Re(c) = 0$ and $\langle b, c \rangle = 0$. By applying an appropriate automorphism of \mathbb{O} we may assume that $b = i, c = j$. Then it remains to show that for any $a \in \mathbb{O}$

$$Re((\bar{a}i)(ja)) = 0.$$

This can be check by a direct computation using a representation $a = u + vl$ with $u, v \in \mathbb{H}$ and the formula (3). Lemma is proved. Q.E.D.

1.2 Octonionic hermitian matrices.

Let us denote

$$\mathbb{O}^m := \{(q_1, \dots, q_m) \mid q_i \in \mathbb{O}\}.$$

For $\xi = (q_1, \dots, q_m) \in \mathbb{O}^m$, $x \in \mathbb{O}$ we will denote by $\xi \cdot x$ the m -tuple $(q_1x, \dots, q_mx) \in \mathbb{O}^m$. Notice that usually we will write elements of \mathbb{O}^m as m -columns rather than rows.

Let us denote by $\mathcal{H}_n(\mathbb{R})$ the space of real symmetric $(n \times n)$ -matrices. The space $\mathcal{H}_n(\mathbb{R})$ is naturally identified with the space of real valued quadratic forms on \mathbb{R}^n .

Let us denote by $\mathcal{H}_m(\mathbb{O})$ the space of octonionic hermitian $(m \times m)$ -matrices. By definition, an $(m \times m)$ -matrix $A = (a_{ij})$ with octonionic entries is called hermitian if $a_{ij} = \overline{a_{ji}}$ for any i, j . For a matrix $A = (a_{ij})$ denote also $A^* := (\overline{a_{ji}})$. We will be mostly interested in octonionic hermitian matrices of size 2. In this case we have the following explicit description

$$\mathcal{H}_2(\mathbb{O}) = \left\{ \begin{bmatrix} a & q \\ \bar{q} & b \end{bmatrix} \mid a, b \in \mathbb{R}, q \in \mathbb{O} \right\}. \quad (5)$$

We have the natural \mathbb{R} -linear map

$$j: \mathcal{H}_2(\mathbb{O}) \rightarrow \mathcal{H}_{16}(\mathbb{R}) \quad (6)$$

which is defined as follows: for any $A \in \mathcal{H}_2(\mathbb{O})$ the value of the quadratic form $j(A)$ on any octonionic 2-column $\xi \in \mathbb{O}^2 \simeq \mathbb{R}^{16}$ is equal $j(A)(\xi) = Re(\xi^* A \xi)$ (note that the bracketing inside the formula is not important due to Lemma 1.1.1(i)). It is easy to see that the map j is injective. Via this map j we will identify $\mathcal{H}_2(\mathbb{O})$ with a subspace of $\mathcal{H}_{16}(\mathbb{R})$.

Let us construct now a linear map

$$\theta: \mathcal{H}_{16}(\mathbb{R}) \rightarrow \mathcal{H}_2(\mathbb{O})$$

such that $\theta \circ j = Id$ and which will be useful later. For any $B \in \mathcal{H}_{16}(\mathbb{R})$ let us denote by b the corresponding quadratic form on $\mathbb{R}^{16} \simeq \mathbb{O}^2$. Define

$$\theta(B) := \frac{1}{16} \left(\frac{\partial^2 b}{\partial \bar{q}_i \partial q_j} \right)_{i,j=1}^2$$

to be the octonionic Hessian of b all of whose entries are replaced by octonionic conjugates (notice the reversed order of the Dirac operators which influences the indexing). Note that the matrix in the right hand side of the last formula is independent of a point in \mathbb{O}^2 .

1.2.1 Lemma. For any $B \in \mathcal{H}_{16}(\mathbb{R})$ and any $\xi \in \mathbb{O}^2$ of the form either $\xi = \begin{bmatrix} a \\ 1 \end{bmatrix}$ or $\xi = \begin{bmatrix} 1 \\ a \end{bmatrix}$, $a \in \mathbb{O}$, one has

$$\theta(B)(\xi) := \text{Re}(\xi^* \theta(B) \xi) = \int_{x \in \mathbb{O}, |x|=1} b(\xi \cdot x) dx$$

where dx is the rotation invariant probability measure on $S^7 = \{x \in \mathbb{O}, |x| = 1\}$.

Proof. Let us denote elements of \mathbb{R}^{16} by 16-tuples $(x_0, x_1, \dots, x_7; y_0, y_1, \dots, y_7)$ where x_i 's correspond to the first octonionic coordinate $q_1 = \sum_{i=0}^7 x_i e_i$, and similarly y_i 's correspond to the second quaternionic coordinate q_2 . By linearity and symmetry considerations it is enough to prove the lemma in the following 2 cases for fixed p, q :

- (1) $b((x, y)) = x_p x_q$;
- (2) $b((x, y)) = x_p y_q$ for any $(x, y) \in \mathbb{O}^2$.

Let us start with case (1). For any $\xi = \begin{bmatrix} a \\ 1 \end{bmatrix} \in \mathbb{O}^2$ one has

$$\int_{x \in \mathbb{O}, |x|=1} b(\xi \cdot x) dx = \int_{S^7} (ax)_p (ax)_q dx = |a|^2 \int_{S^7} x_p x_q dx.$$

The last integral vanishes for $p \neq q$ and equals to $\frac{|a|^2}{8}$ for $p = q$. On the other hand if $p \neq q$ $\left(\frac{\partial^2 b}{\partial \bar{q}_i \partial q_j}\right) = 0$, thus lemma is proved in this case. If $p = q$ then

$$\left(\frac{\partial^2 b}{\partial \bar{q}_i \partial q_j}\right) = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}.$$

Hence

$$\theta(B)(\xi) = \frac{1}{16} \text{Re} \left([\bar{a}, 1] \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ 1 \end{bmatrix} \right) = \frac{|a|^2}{8} = \int_{x \in \mathbb{O}, |x|=1} b(\xi \cdot x) dx.$$

Let us consider case (2). In this case

$$\left(\frac{\partial^2 b}{\partial \bar{q}_i \partial q_j}\right) = \begin{bmatrix} 0 & e_p \bar{e}_q \\ e_q \bar{e}_p & 0 \end{bmatrix}.$$

Let first $\xi = \begin{bmatrix} a \\ 1 \end{bmatrix}$. Then

$$\theta(B)(\xi) = \frac{1}{16} \text{Re} \left([\bar{a}, 1] \begin{bmatrix} 0 & e_p \bar{e}_q \\ e_q \bar{e}_p & 0 \end{bmatrix} \begin{bmatrix} a \\ 1 \end{bmatrix} \right) = \frac{1}{8} \text{Re}(e_q \bar{e}_p a).$$

On the other hand

$$\begin{aligned} \int_{x \in \mathbb{O}, |x|=1} b(\xi \cdot x) d(x) &= \int_{S^7} (ax)_p x_q d(x) = \int_{S^7} \text{Re}(\bar{e}_p ax) \cdot x_q dx = \\ &= \int_{S^7} \text{Re} \left(\bar{e}_p a \left(\sum_{s=0}^7 x_s e_s \right) \right) x_q dx = \int_{S^7} \text{Re}(\bar{e}_p a e_q) x_q^2 = \frac{1}{8} \text{Re}(e_q \bar{e}_p a) = \theta(B)(\xi). \end{aligned}$$

Similarly one considers the case $\xi = \begin{bmatrix} 1 \\ a \end{bmatrix}$. Lemma is proved. Q.E.D.

1.2.2 Corollary.

$$\theta \circ j = Id.$$

Proof. It is easy to see that any matrix $A = \begin{bmatrix} \alpha & q \\ \bar{q} & \beta \end{bmatrix} \in \mathcal{H}_2(\mathbb{O})$ is uniquely determined by the products $Re(\xi^* A \xi)$ where $\xi = \begin{bmatrix} a \\ 1 \end{bmatrix} \in \mathbb{O}^2$. By Lemma 1.2.1 one has

$$\begin{aligned} ((\theta \circ j)(A))(\xi) &= \int_{S^7} (jA)(\xi \cdot x) dx = \int_{S^7} Re \left([\bar{ax}, \bar{x}] \begin{bmatrix} \alpha & q \\ \bar{q} & \beta \end{bmatrix} \begin{bmatrix} ax \\ x \end{bmatrix} \right) dx = \\ &= \int_{S^7} (\alpha|a|^2 + \beta + 2Re((\bar{ax})qx)) dx \stackrel{\text{Lemma 1.1.1(v)}}{=} \alpha|a|^2 + \beta + 2Re(q\bar{a}) = \\ &= Re \left([\bar{a}, 1] \begin{bmatrix} \alpha & q \\ \bar{q} & \beta \end{bmatrix} \begin{bmatrix} a \\ 1 \end{bmatrix} \right) = Re(\xi^* A \xi). \end{aligned}$$

Corollary is proved. Q.E.D.

1.2.3 Definition. Let $A \in \mathcal{H}_2(\mathbb{O})$. A is called *positive definite* (resp. *non-negative definite*) if for any 2-column $\xi \in \mathbb{O}^2 \setminus \{0\}$

$$Re(\xi^* A \xi) > 0 \text{ (resp. } Re(\xi^* A \xi) \geq 0 \text{)}.$$

For a positive definite (resp. non-negative definite) matrix A one writes as usual $A > 0$ (resp. $A \geq 0$).

On the class of octonionic hermitian (2×2) -matrices there is a nice notion of determinant which is defined by

$$\det \left(\begin{bmatrix} a & q \\ \bar{q} & b \end{bmatrix} \right) = ab - |q|^2. \quad (7)$$

1.2.4 Remark. It turns out that a nice notion of determinant does exist also on octonionic hermitian matrices of size 3, see e.g. Section 3.4 of [18]. Note also that a nice determinant does exist for *quaternionic* hermitian matrices of any size: see the survey [17], the article [29], and for applications to quaternionic plurisubharmonic functions see [6], [7], [11], [13].

The following result is a version of the Sylvester criterion for octonionic matrices of size two.

1.2.5 Proposition. Let $A = \begin{bmatrix} a & q \\ \bar{q} & b \end{bmatrix} \in \mathcal{H}_2(\mathbb{O})$. Then $A > 0$ if and only if $a > 0$ and $\det A > 0$.

Proof. Let $\xi = \begin{bmatrix} x \\ y \end{bmatrix} \in \mathcal{H}_2(\mathbb{O})$. Then

$$Re(\xi^* A \xi) = a|x|^2 + b|y|^2 + 2Re(qy\bar{x}). \quad (8)$$

Assume first that $A > 0$. Taking $x = 1, y = 0$ we get $a > 0$. Next substituting into (8) $y = t|q|q^{-1}x$ where $t \in \mathbb{R}$ we get for any $x \in \mathbb{O} \setminus \{0\}$ and any $t \in \mathbb{R}$

$$|x|^2(a + bt^2 + 2t|q|) > 0.$$

hence $0 < ab - |q|^2 = \det A$.

Conversely assume that $a > 0, \det A > 0$. Since $|Re(uv)| \leq |u| \cdot |v|$, (8) implies that

$$Re(\xi^* A \xi) \geq a|x|^2 + b|y|^2 - 2|q| \cdot |x| \cdot |y|.$$

Our assumptions imply that the last expression is positive provided $(x, y) \neq 0$. Q.E.D.

Now we will introduce the notion of *mixed determinant* of two octonionic hermitian matrices in analogy to the classical real case (see e.g. [45]). First observe that the determinant $\det \mathcal{H}_2(\mathbb{O}) \rightarrow \mathbb{R}$ is a homogeneous polynomial of degree 2 on the real vector space $\mathcal{H}_2(\mathbb{O})$. Hence it admits a unique polarization: a bilinear symmetric map

$$D: \mathcal{H}_2(\mathbb{O}) \times \mathcal{H}_2(\mathbb{O}) \rightarrow \mathbb{R}$$

such that $D(A, A) = \det A$ for any $A \in \mathcal{H}_2(\mathbb{O})$. This map D is called the mixed determinant. By the abuse of notation it will be denoted again by \det . Explicitly if $A = (a_{ij})_{i,j=1}^2$, $B = (b_{ij})_{i,j=1}^2$ are octonionic hermitian then

$$\det(A, B) = \frac{1}{2} (a_{11}b_{22} + a_{22}b_{11} - 2Re(a_{12}b_{21})). \quad (9)$$

1.2.6 Lemma. *If $A, B \in \mathcal{H}_2(\mathbb{O})$ are positive (resp. non-negative) definite then*

$$\det(A, B) > 0 \text{ (resp. } \det(A, B) \geq 0 \text{)}.$$

Proof. Let us assume that A, B are positive definite. Then by Proposition 1.2.5 we have

$$\begin{aligned} a_{11} > 0, a_{22} > 0, |a_{12}| &< \sqrt{a_{11}a_{22}}, \\ b_{11} > 0, b_{22} > 0, |b_{12}| &< \sqrt{b_{11}b_{22}}. \end{aligned}$$

These inequalities together with the Cauchy-Schwarz inequality imply

$$Re(a_{12}b_{21}) \leq |a_{12}| \cdot |b_{21}| < \sqrt{a_{11}a_{22}b_{11}b_{22}}. \quad (10)$$

Substituting (10) into (9) we get

$$\det(A, B) > \frac{1}{2} \left(a_{11}b_{22} + a_{22}b_{11} - 2\sqrt{a_{11}a_{22}b_{11}b_{22}} \right) \geq 0$$

where the last estimate is the arithmetic-geometric mean inequality. Thus $\det(A, B) > 0$ for positive definite matrices A, B . For non-negative definite matrices the result follows by going to the limit. Q.E.D.

1.2.7 Remark. Also one can prove the following version of the Aleksandrov inequality for mixed determinants [2]: The mixed determinant $\det(\cdot, \cdot)$ is a non-degenerate quadratic form on the real vector space $\mathcal{H}_2(\mathbb{O})$; its signature type has one plus and the rest are minuses. Consequently, if $A > 0$ then for any $B \in \mathcal{H}_2(\mathbb{O})$ one has

$$\det(A, B)^2 \geq \det A \cdot \det B$$

and the equality is achieved if and only if A is proportional to B with a real coefficient.

We do not present a detailed proof of this result since we are not going to use it. Notice only that this result can be deduced formally from the corresponding result for quaternionic matrices proved in [6]. Indeed all the entries of A and B together contain at most two non-real octonions, hence the field generated them is associative by Lemma 1.1.1(iv).

1.3 Octonionic projective line \mathbb{OP}^1 .

In this section we remind the definition and basic properties of the octonionic projective line \mathbb{OP}^1 .

Let us define an equivalence relation \sim on the unit sphere $S^{15} \subset \mathbb{O}^2$ by saying $\xi \sim \eta$ if and only if

$$\xi\xi^* = \eta\eta^*$$

where we write ξ and η as columns, thus $\xi\xi^*$ and $\eta\eta^*$ are octonionic (2×2) -matrices. It is easy to see that $\begin{bmatrix} x \\ y \end{bmatrix} \sim \begin{bmatrix} xy^{-1} \\ 1 \end{bmatrix}$ if $y \neq 0$, and $\begin{bmatrix} x \\ y \end{bmatrix} \sim \begin{bmatrix} 1 \\ yx^{-1} \end{bmatrix}$ if $x \neq 0$. The quotient of S^{15} by this equivalence relation is called the octonionic projective line \mathbb{OP}^1 .

1.3.1 Remark. If in the above construction one replaces \mathbb{O} by \mathbb{R} , \mathbb{C} , or \mathbb{H} , one get the usual projective lines \mathbb{RP}^1 , \mathbb{CP}^1 , \mathbb{HP}^1 respectively.

\mathbb{OP}^1 has a natural smooth structure, and it is diffeomorphic to the standard sphere S^8 . The fibers of the quotient map $S^{15} \rightarrow \mathbb{OP}^1$ are the spheres S^7 . This map is called the octonionic Hopf fibration.

1.4 The group $SL_2(\mathbb{O})$.

We discuss in this section the definition and basic properties of the group $SL_2(\mathbb{O})$. We refer to [46] for the proofs and further details.

An octonionic (2×2) -matrix is called *traceless* if the sum of its diagonal elements is equal to zero. Every (2×2) -matrix A with octonionic entries defines an \mathbb{R} -linear operator on \mathbb{O}^2 by $\xi \mapsto A \cdot \xi$. However the space of such operators is not closed under the commutator due to the lack of associativity. One denotes by $sl_2(\mathbb{O})$ the Lie subalgebra of $gl_{16}(\mathbb{R})$ generated by of \mathbb{R} -linear operators on $\mathbb{O}^2 \simeq \mathbb{R}^{16}$ determined by all traceless octonionic matrices. This Lie algebra $sl_2(\mathbb{O})$ turns out to be semi-simple [46] (see Theorem 1.4.1 below for details). But any semi-simple Lie subalgebra of an algebraic group is a Lie algebra of a closed algebraic subgroup (see e.g. [43], Ch. 3, §3.3). In our case this subgroup is denoted by $SL_2(\mathbb{O}) \subset GL(16, \mathbb{R})$.

1.4.1 Theorem ([46]). (i) The Lie algebra $sl_2(\mathbb{O})$ is isomorphic to the Lie algebra $so(9, 1)$.
(ii) The Lie group $SL_2(\mathbb{O})$ is isomorphic to the group $Spin(9, 1)$ (which is the universal covering of the identity component of the pseudo-orthogonal group $O(9, 1)$).

1.4.2 Remark. A maximal compact subgroup of $SL_2(\mathbb{O}) \simeq Spin(9, 1)$ is isomorphic to the group $Spin(9)$ which is the universal covering of the special orthogonal group $SO(9)$.

Both $sl_2(\mathbb{O})$ and $SL_2(\mathbb{O})$ come with their *fundamental representations* on \mathbb{O}^2 . Moreover the Lie algebra $sl_2(\mathbb{O})$ acts on the space $\mathcal{H}_2(\mathbb{O})$. This action is uniquely characterized by the following property (see [46] for the details): if A is a traceless matrix it acts by $A: X \mapsto -A^*X - XA$. Since the group $SL_2(\mathbb{O})$ is connected and simply connected, this representation of $sl_2(\mathbb{O})$ integrates to a representation of the group $SL_2(\mathbb{O})$ on $\mathcal{H}_2(\mathbb{O})$.

1.4.3 Proposition. The group $SL_2(\mathbb{O})$ preserves the cone of positive definite octonionic hermitian matrices.

Proof. Let us denote by \mathcal{K} the open cone of positive definite matrices in $\mathcal{H}_2(\mathbb{O})$. Let us denote by $\bar{\mathcal{K}}$ the closure of \mathcal{K} , namely the closed cone of non-negative definite matrices. The boundary $\partial\bar{\mathcal{K}}$ is a hypersurface in $\mathcal{H}_2(\mathbb{O})$ which is smooth at every point except of 0.

In order to prove the proposition it is enough to prove the infinitesimal version of it as follows. Let us consider any element $D \in sl_2(\mathbb{O})$ from the Lie algebra. It induces a vector field on $\mathcal{H}_2(\mathbb{O})$ via its action: $X \mapsto D(X)$. In order to show that the one-parametric subgroup in $SL_2(\mathbb{O})$ generated by D preserves the cone \mathcal{K} it is enough to check that at any point $X \in \partial\bar{\mathcal{K}}$ the vector $D(X)$ is not directed outside of the domain $\bar{\mathcal{K}}$ (i.e. looks inside or tangent to the boundary $\partial\bar{\mathcal{K}}$) when X is a smooth point of the boundary, and vanishes when X is a singular point of the boundary. Clearly $D(0) = 0$, and 0 is the only singular point of the boundary. Hence we may assume that X is a smooth point of the boundary. We are going to show that the vector $D(X)$ is in fact tangent to $\partial\bar{\mathcal{K}}$. Since $\partial\bar{\mathcal{K}} = \{A \geq 0 \mid \det A = 0\}$, this follows from the fact that the group $SL_2(\mathbb{O})$ preserves the determinant of octonionic hermitian matrices (see e.g. [18], p. 177). Proposition is proved. Q.E.D.

The following two lemmas are essentially contained in [38], but we would like to present a proof for the sake of completeness.

1.4.4 Lemma ([38]). The linear map $\mathbb{O}^2 \otimes_{\mathbb{R}} \mathbb{O}^2 \rightarrow \mathcal{H}_2(\mathbb{O})$ given by

$$\xi \otimes \eta \mapsto \xi \cdot \eta^* + \eta \cdot \xi^*$$

is $SL_2(\mathbb{O})$ -equivariant.

Proof. It is enough to prove the equivariance with respect to the action of the Lie algebra $sl_2(\mathbb{O})$. In fact it is enough to show that for any octonionic traceless (2×2) -matrix M and any $\xi \in \mathbb{O}^2$ one has

$$(M\xi) \cdot \xi^* + \xi \cdot (M\xi)^* = M \cdot (\xi\xi^*) + (\xi\xi^*) \cdot M^*. \quad (11)$$

This equality is proved by a straightforward computation using Lemma 1.1.1. Q.E.D.

1.4.5 Lemma ([38]). (i) The group $SL_2(\mathbb{O})$ acts naturally on \mathbb{OP}^1 , namely for any $\phi \in SL_2(\mathbb{O})$ and any $L \in \mathbb{OP}^1$ the subspace $\phi(L)$ is an octonionic projective line.

(ii) For any $L \in \mathbb{OP}^1$ and any $\phi \in SL_2(\mathbb{O})$ the restriction

$$\phi|_L: L \rightarrow \phi(L)$$

is a conformal linear map.

Proof. First let us show that if $\xi, \eta \in \mathbb{O}^2$ have norm 1 and $\xi \sim \eta$ then $|\phi(\xi)| = |\phi(\eta)|$. Observe that for any $v \in \mathbb{O}^2$ one has

$$|v|^2 = v^*v = Tr(vv^*)$$

where Tr denotes the sum of diagonal elements of a matrix. Then we have

$$\begin{aligned} |\phi(\xi)|^2 &= Tr(\phi(\xi) \cdot \phi(\xi)^*) \stackrel{\text{by Lemma 1.4.4}}{=} Tr(\phi(\xi \cdot \xi^*)) = \\ &Tr(\phi(\eta \cdot \eta^*)) \stackrel{\text{by Lemma 1.4.4}}{=} Tr(\phi(\eta) \cdot \phi(\eta)^*) = |\phi(\eta)|^2. \end{aligned}$$

Thus $|\phi(\xi)| = |\phi(\eta)|$. Then in order to prove both parts of the lemma it remains to show that for any norm 1 vectors $\xi, \eta \in \mathbb{O}^2$ such that $\xi \sim \eta$ (i.e. $\xi\xi^* = \eta\eta^*$) one has $\phi(\xi) \sim \phi(\eta)$, i.e. $\phi(\xi) \cdot \phi(\xi)^* = \phi(\eta) \cdot \phi(\eta)^*$. But applying Lemma 1.4.4 twice we get

$$\phi(\xi) \cdot \phi(\xi)^* = \phi(\xi\xi^*) = \phi(\eta\eta^*) = \phi(\eta) \cdot \phi(\eta)^*.$$

Q.E.D.

We immediately deduce the following corollary.

1.4.6 Corollary. The octonionic Hopf map $S^{15} \rightarrow \mathbb{OP}^1$ is $Spin(9)$ -equivariant.

The following lemma is well known [23].

1.4.7 Lemma. The group $Spin(9)$ acts transitively on the unit sphere S^{15} , and hence on \mathbb{OP}^1 .

1.5 Further properties of the octonionic Hessian.

1.5.1 Proposition. Let $f: \mathbb{O}^2 \rightarrow \mathbb{R}$ be a C^2 -smooth function. Let $A \in SL_2(\mathbb{O})$. Then

$$\left(\frac{\partial^2}{\partial q_i \partial \bar{q}_j} (f(Aq)) \right) = A \left(\frac{\partial^2 f}{\partial q_i \partial \bar{q}_j} (Aq) \right)$$

where A in the right hand side denotes the induced action of A on $\mathcal{H}_2(\mathbb{O})$.

Proof. By translation it is enough to check the above equality at $q = 0$. Moreover we may and will assume that f is a quadratic form. Thus the proposition becomes equivalent to the the following lemma.

1.5.2 Lemma. The map $\theta: \mathcal{H}_{16}(\mathbb{R}) \rightarrow \mathcal{H}_2(\mathbb{O})$ is $SL_2(\mathbb{O})$ -equivariant.

Proof. It is enough to prove this proposition infinitesimally, i.e. for the action of the Lie algebra $sl_2(\mathbb{O})$. Moreover it is sufficient to check it for a set of generators, say for traceless (2×2) -octonionic matrices. Let us fix such a traceless matrix A . We have to show that for any $b \in \mathcal{H}_{16}(\mathbb{R})$ one has

$$\theta(A(b)) = A(\theta(b)). \quad (12)$$

It is enough to show that for any $\xi = \begin{bmatrix} p \\ 1 \end{bmatrix} \in \mathbb{O}^2$ one has

$$Re(\xi^* \theta(A(b)) \xi) = Re(\xi^* A(\theta(b)) \xi). \quad (13)$$

By Lemma 1.2.1 the right hand side of (13) can be rewritten as

$$- Re(\xi^* (A^* \theta(b)) \xi + \xi^* (\theta(b) A) \xi) = -2 Re(\xi^* (A^* \theta(b)) \xi). \quad (14)$$

By Lemma 1.2.1 the left hand side of (13) can be rewritten as follows:

$$-2 \int_{x \in S^7} b((A\xi)x, \xi x) dx = -\frac{d}{d\tau} \Big|_0 \int_{x \in S^7} b((A\xi + \tau\xi)x) dx = \quad (15)$$

$$-\frac{d}{d\tau} \Big|_0 \theta(b)(A\xi + \tau\xi) = -\frac{d}{d\tau} \Big|_0 Re((A\xi + \tau\xi)^* \theta(b)(A\xi + \tau\xi)) = \quad (16)$$

$$-2 Re((\xi^* A^*) \theta(b) \xi). \quad (17)$$

Substituting (14) and (17) into (13) we see that (13) becomes equivalent to

$$Re(\xi^* (A^* \theta(b)) \xi) = Re((\xi^* A^*) \theta(b) \xi). \quad (18)$$

Both sides of the above equality are linear with respect to $\theta(b) \in \mathcal{H}_2(\mathbb{O})$. Obviously the equality is satisfied when $\theta(b)$ has real entries. Thus we may assume that $\theta(b) = \begin{bmatrix} 0 & q \\ \bar{q} & 0 \end{bmatrix}$.

Let us denote also $A^* = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. In this notation (18) becomes

$$Re\left(\begin{bmatrix} \bar{p} & 1 \end{bmatrix} \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 & q \\ \bar{q} & 0 \end{bmatrix} \right) \begin{bmatrix} p \\ 1 \end{bmatrix} \right) = Re\left(\left(\begin{bmatrix} \bar{p} & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) \begin{bmatrix} 0 & q \\ \bar{q} & 0 \end{bmatrix} \begin{bmatrix} p \\ 1 \end{bmatrix} \right) \quad (19)$$

By a direct computation the right hand side of (19) is equal to

$$Re((\bar{p}b)(\bar{q}p)) + Re(\bar{p}aq + cq + d\bar{q}p). \quad (20)$$

The left hand side of (19) is equal to

$$|p|^2 Re(b\bar{q}) + Re(\bar{p}aq + cq + d\bar{q}p). \quad (21)$$

Comparing (20) and (21), it remains to prove that for any $b, p, q \in \mathbb{O}$ one has

$$Re((\bar{p}b)(qp)) = |p|^2 Re(bq). \quad (22)$$

But this is exactly Lemma 1.1.1. Hence Lemma 1.5.2 is proved. Q.E.D.

1.5.3 Remark. Similarly one can show that the imbedding $j: \mathcal{H}_2(\mathbb{O}) \hookrightarrow \mathcal{H}_{16}(\mathbb{R})$ (see (6) in Section 1.2) is also $SL_2(\mathbb{O})$ -equivariant.

2 Octonionic Radon transform.

2.1.1 Definition. An *affine octonionic line* in \mathbb{O}^2 is any translate of an octonionic line from \mathbb{OP}^1 .

The manifold of all affine octonionic lines in \mathbb{O}^2 will be denoted by \mathcal{AOP}^1 . It is a homogeneous space for the group $\mathbb{O}^2 \rtimes Spin(9)$ (we denote in this way the semi-direct product of the group \mathbb{O}^2 of translations of \mathbb{O}^2 , and of the group $Spin(9)$ of certain linear transformations).

Let us define the Radon transform operator

$$R: C_0^\infty(\mathbb{O}^2) \rightarrow C_0^\infty(\mathcal{AOP}^1) \quad (23)$$

by $(Rf)(E) = \int_{q \in E} f(q) dq$ where dq is the Lebesgue measure induced by the standard Euclidean metric on \mathbb{O}^2 .

2.1.2 Proposition. *The octonionic Radon transform (23) is injective.*

Proof. We will just present the inversion formula completely analogous to the complex Radon transform (see [28]; see also appendix in [11] for the quaternionic case). For any point $q \in \mathbb{O}^2$ let \mathcal{P}_q denote the manifold of affine octonionic lines passing through q . Clearly $\mathcal{P}_q \simeq \mathbb{OP}^1$. For $E \in \mathcal{AOP}^1$ let us denote by E^\perp the octonionic line orthogonal to E and passing through the origin 0.

Let us define the operator

$$\mathcal{D}: C^\infty(\mathcal{AOP}^1) \rightarrow C^\infty(\mathbb{O}^2)$$

as follows. Let $g \in C^\infty(\mathcal{AOP}^1)$. Set

$$\mathcal{D}g(q) := \int_{E \in \mathcal{P}_q} (\Delta_{E^\perp})^4 g(E + w) dE,$$

where Δ_{E^\perp} denotes the (8-dimensional) Laplacian with respect to $w \in E^\perp$, and the integration is with respect the Haar measure on \mathcal{P}_q invariant under the action of $Spin(9)$.

2.1.3 Claim. *For any smooth rapidly decreasing function f of \mathbb{O}^2*

$$\mathcal{D}(Rf) = c \cdot f,$$

where c is a non-zero constant.

It is sufficient to check this claim pointwise, say at 0. The operators R and \mathcal{D} commute with translations and the action of the group $Spin(9)$. Then $\mathcal{D}(Rf)(0)$ defines a distribution which is invariant with respect to the action of $Spin(9)$. Moreover it is easy to check that this distribution is homogeneous of degree -16 (exactly as the delta-function at 0). Since the group $Spin(9)$ acts transitively on the unit sphere S^{15} , there is at most one dimensional space of $Spin(9)$ -invariant distributions homogeneous of degree -16 . Hence they must be proportional to the delta-function at 0. Thus $\mathcal{D}(Rf) = c \cdot f$ for some constant c . To see

that $c \neq 0$ it is sufficient to check it by an explicit computation for the function $f(q) = \exp(-|q|^2/2)$. Q.E.D.

For an affine line $L \in \mathcal{AOP}^1$ let us denote by δ_L the generalized function given by

$$\delta_L(\mu) = \int_L \mu$$

for any infinitely smooth compactly supported measure μ .

2.1.4 Corollary. *The \mathbb{C} -linear span of δ -functions of affine octonionic lines is dense in the weak topology in the space $C^{-\infty}(\mathbb{O}^2)$ of generalized functions.*

Proof. This follows immediately from Proposition 2.1.2 and the Hahn-Banach theorem. Q.E.D.

3 Octonionic plurisubharmonic functions and their properties.

3.1 Octonionic plurisubharmonic functions.

Let us remind few standard definitions. Recall that a real valued function f is called upper semi-continuous if

$$f(x_0) \geq \limsup_{x \rightarrow x_0} f(x)$$

for any point x_0 .

3.1.1 Definition. Let $U \subset \mathbb{R}^n$ be an open subset. A function $f: U \rightarrow \mathbb{R} \cup \{-\infty\}$ is called *subharmonic* if

- (i) f is upper semi-continuous;
- (ii) for any point $x_0 \in U$ and any sphere $S \subset U$ centered at x_0 one has

$$f(x_0) \leq \int_S f(x) dx \tag{24}$$

where dx is the probability rotation invariant measure on the sphere S .

According to this definition the function which is identically equal to $-\infty$ is subharmonic.

3.1.2 Remark. (1) The integral in (24) is understood in the following sense. Any upper semi-continuous function is Borel measurable and locally bounded from above. Hence $f|_S$ is measurable and bounded from above on S . Hence the integral $\int_S f(x) dx \in \mathbb{R} \cup \{-\infty\}$ is well defined.

(2) It is well known that for a subharmonic function $f \not\equiv -\infty$ the integral $\int_S f(x) dx$ is always finite.

(3) It is well known (see e.g. [34], Corollary 3.2.8) that any subharmonic function $f \not\equiv -\infty$ is locally integrable. In particular $f > -\infty$ almost everywhere. There exists the following characterization of subharmonic functions (see e.g. [34], Theorem 3.2.11). If a function

$f: U \rightarrow \mathbb{R} \cup \{-\infty\}$, $f \not\equiv -\infty$, is subharmonic then $\Delta f \geq 0$ in sense of generalized functions, where $\Delta = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$ is the usual Laplacian on \mathbb{R}^n . (Notice that in order to define Δf we use the fact that f is locally integrable.) Conversely if U is a generalized function, $\Delta U \geq 0$, then U is defined by a unique subharmonic function $u \not\equiv -\infty$. In particular twice continuously differentiable function f is subharmonic if and only if $\Delta f \geq 0$ pointwise.

Let $\Omega \subset \mathbb{O}^2$ be an open subset.

3.1.3 Definition. Let $f: \Omega \rightarrow \mathbb{R} \cup \{-\infty\}$. The function f is called *octonionic plurisubharmonic* if

- (i) f is upper semi-continuous;
- (ii) the restriction of f to any affine octonionic line $L \in \mathcal{AOP}^1$ is subharmonic.

We will denote by $P(\Omega)$ the class of all octonionic plurisubharmonic functions in Ω .

3.1.4 Example. Any convex function on $\mathbb{O}^2 \simeq \mathbb{R}^{16}$ is octonionic subharmonic.

3.1.5 Proposition. *The class of octonionic plurisubharmonic functions is invariant under the group $\mathbb{O}^2 \rtimes SL_2(\mathbb{O})$.*

Proof. This follows immediately from Lemma 1.4.5. Q.E.D.

3.1.6 Proposition. *Any octonionic plurisubharmonic function is subharmonic.*

Proof. Let f be an octonionic plurisubharmonic function. We have to show that for any x_0 and any sphere S centered in x_0 (and both contained in our domain) one has

$$f(x_0) \leq \int_S f(x) dx.$$

Without loss of generality we may assume that $x_0 = 0$ and S has radius 1. Then we have the equality

$$\int_S f(x) dx = \int_{\mathbb{OP}^1} \left(\int_{y \in S(L)} f(y) dy \right) dL \quad (25)$$

where $S(L)$ denotes the unit sphere in L , and dL is the only $Spin(9)$ -invariant probability measure on \mathbb{OP}^1 . The equality (25) follows from the uniqueness of $Spin(9)$ -invariant probability measure on S .

Then we have

$$\int_S f(x) dx = \int_{\mathbb{OP}^1} \left(\int_{y \in S(L)} f(y) dy \right) dL \geq \int_{\mathbb{OP}^1} f(x_0) dL = f(x_0).$$

Q.E.D.

3.1.7 Corollary. *Any octonionic plurisubharmonic function $\not\equiv -\infty$ is locally integrable.*

Proof. Since any subharmonic function $\not\equiv -\infty$ is locally integrable (see e.g. [34], Corollary 3.2.8) the result follows from Proposition 3.1.6. Q.E.D.

3.1.8 Proposition. *Let $f \in C^2(\Omega)$. Then f is octonionic plurisubharmonic if and only if the matrix $\left(\frac{\partial^2 f}{\partial q_i \partial \bar{q}_j}\right)$ is non-negative definite pointwise.*

Proof. Let us start with the following elementary observation. Let b be a quadratic form on a Euclidean space L . Then

$$Tr(b) = \dim L \cdot \int_{y \in S(L)} b(y) dy \quad (26)$$

where $S(L)$ denotes the unit sphere of L , and dy is the rotation invariant probability measure on $S(L)$.

Let us fix now $z \in \Omega$. Let us denote by b_z the usual (real) Hessian of the function f at the point z . Let us fix also an affine octonionic line L passing through z . We have to show that $\Delta_L(f|_L)|_z \geq 0$ where Δ_L denotes the Laplacian on the line L . But

$$\Delta_L(f|_L)|_z = 2Tr(b_z|_L).$$

Using this and (26) we get

$$\Delta_L(f|_L)|_z = 16 \int_{S(L)} b_z(y) dy. \quad (27)$$

We may and will assume that $L = z + \{\xi \cdot x \mid x \in \mathbb{O}\}$ for some fixed $\xi = \begin{bmatrix} a \\ 1 \end{bmatrix} \in \mathbb{O}^2$. Then

$$\begin{aligned} \Delta_L(f|_L)|_z &= \frac{16}{|\xi|^2} \int_{x \in S^7} b_z(\xi \cdot x) dx \stackrel{\text{Lemma 1.2.1}}{=} \\ &\quad \frac{1}{|\xi|^2} Re \left(\xi^* \left(\frac{\partial^2 f(z)}{\partial q_i \partial \bar{q}_j} \right) \xi \right). \end{aligned}$$

Thus we have shown that

$$\Delta_L(f|_L)|_z = \frac{1}{|\xi|^2} Re \left(\xi^* \left(\frac{\partial^2 f(z)}{\partial q_i \partial \bar{q}_j} \right) \xi \right). \quad (28)$$

The last identity obviously implies the proposition. Q.E.D.

Let dq denote the standard Lebesgue measure on \mathbb{O}^2 . This choice will allow us identify functions with measures via $u \mapsto u \cdot dq$. Thus we will not distinguish these two notions.

A *generalized function* with values in $\mathcal{H}_2(\mathbb{O})$, by definition, is a continuous linear \mathbb{R} -valued functional on the space $C_0^\infty(\Omega, \mathcal{H}_2(\mathbb{O}))$ of infinitely smooth compactly supported functions on Ω with values in $\mathcal{H}_2(\mathbb{O})$. This space will be denoted by $C^{-\infty}(\Omega, \mathcal{H}_2(\mathbb{O}))$. It is equipped with the weak topology. We have an imbedding of the space $L_{loc}^1(\Omega, \mathcal{H}_2(\mathbb{O}))$ of locally integrable $\mathcal{H}_2(\mathbb{O})$ -valued functions into $C^{-\infty}(\Omega, \mathcal{H}_2(\mathbb{O}))$ which is given by

$$M \mapsto [\Phi \mapsto \int_{\Omega} Re(Tr(M \cdot \Phi)) dq]$$

where Tr denotes the sum of diagonal elements. (note also that any octonionic matrices A and B satisfy $Re(Tr(AB)) = Re(Tr(BA))$).

3.1.9 Definition. Let $M \in C^{-\infty}(\Omega, \mathcal{H}_2(\mathbb{O}))$. We say that M is *non-negative* if for any infinitely smooth compactly supported function $\Phi: \Omega \rightarrow \mathcal{H}_2(\mathbb{O})$ such that $\Phi(q) \geq 0$ for any $q \in \Omega$ one has

$$M(\Phi) \geq 0.$$

3.1.10 Remark. If M is a continuous $\mathcal{H}_2(\mathbb{O})$ -valued function then it is non-negative in sense of generalized functions if and only if it is pointwise non-negative. This follows from the observation that a matrix $A \in \mathcal{H}_2(\mathbb{O})$ is non-negative definite if and only if for any non-negative definite matrix B one has $\text{Re}(\text{Tr}(AB)) \geq 0$.

3.1.11 Proposition. (i) *A linear combination of octonionic plurisubharmonic functions with positive coefficients is octonionic plurisubharmonic.*

(ii) *Maximum of two octonionic plurisubharmonic functions is octonionic plurisubharmonic.*

Proof. This follows immediately from the corresponding properties of subharmonic functions. Q.E.D.

Let us fix a sequence $\{\delta_n\}$ of smooth functions approximating the δ -function at the origin

0. More precisely for any $n \in \mathbb{N}$ we fix a function $\delta_n: \mathbb{O}^2 \rightarrow \mathbb{R}$ which satisfies:

- (i) $\delta_n \in C^\infty(\mathbb{O}^2)$;
- (ii) $\delta_n \geq 0$;
- (iii) $\int_{\mathbb{O}^2} \delta_n(x) dx = 1$;
- (iv) the support $\text{supp}(\delta_n)$ is contained in the ball of radius $\frac{1}{n}$ centered at the origin.

3.1.12 Proposition. Let $f \in P(\Omega)$, $f \not\equiv -\infty$ (in particular $f \in L_{loc}^1(\Omega)$ by Corollary 3.1.7).

(i) *Then $f * \delta_n$ is infinitely smooth octonionic plurisubharmonic function and $f * \delta_n \rightarrow f$ in L_{loc}^1 .*

(ii) *If moreover $f \in P(\Omega) \cap C(\Omega)$ then $f * \delta_n \rightarrow f$ uniformly on compact subsets of Ω .*

Proof. Part (ii) is obvious. Let us prove part (i). It is standard (and easy to see) that for any $g \in L_{loc}^1$, $g * \delta_n \rightarrow g$ weakly in sense of measures on every compact subset of Ω , in particular in the sense of generalized functions. Observe now that in our situation the sequence $\{f * \delta_n\}$ is a sequence of subharmonic functions (by Proposition 3.1.6) with a uniform upper bound on every compact subset of Ω . Since this sequence converges to f in sense of distributions, it convergence to f in L_{loc}^1 by the general result on subharmonic functions ([34], Theorem 3.2.12). Proposition is proved. Q.E.D.

We will need the following well known fact on subharmonic functions (see e.g. [34], Theorem 3.2.12).

3.1.13 Lemma. Let $U \subset \mathbb{R}^n$ be an open connected subset. Let $\{u_j\}$ be a sequence of subharmonic functions in U which have a uniform upper bound on every compact subset of U .

(i) *Then either $\{u_j\} \rightarrow -\infty$ uniformly on every compact subset of U , or else there is a subsequence $\{u_{j_k}\}$ which converges in $L_{loc}^1(U)$.*

(ii) *If $u_j \not\equiv -\infty$ for every j , and $\{u_j\}$ converges to $F \in C^{-\infty}(U)$ in the sense of generalized functions, then F is given by a subharmonic function $f \not\equiv -\infty$ and $u_j \rightarrow f$ in $L_{loc}^1(U)$.*

Proof. See [34], Theorem 3.2.12. Q.E.D.

3.1.14 Proposition. *Let $\Omega \subset \mathbb{O}^2$ be an open subset. Let $f: \Omega \rightarrow \mathbb{R} \cup \{-\infty\}$ be a function. Assume that for every $\phi \in \mathbb{O}^2 \rtimes Spin(9)$ the function $f \circ \phi$ is subharmonic in $\phi^{-1}(\Omega)$. Then f is octonionic plurisubharmonic.*

Proof. The proof is an easy modification of the proof of Theorem 4.1.7 in [34]. Let $z = (z_1, z_2) \in \Omega$. The the function $f(z_1 + w_1, z_2 + \varepsilon w_2)$ is subharmonic in w by hypothesis for small $\varepsilon > 0$. Hence

$$f(z) \leq \int_{|\zeta|=1} f(z_1 + \zeta_1, z_2 + \varepsilon \zeta_2) d\zeta.$$

Since f is upper semi-continuous and locally bounded from above, the Fatou lemma implies as $\varepsilon \rightarrow 0$ that

$$f(z) \leq \int_{|\zeta|=1} f(z_1 + \zeta_1, z_2) d\zeta.$$

The last inequality and Theorem 3.2.3 of [34] imply that the function $z_1 \mapsto f(z_1, z_2)$ is subharmonic. The subharmonicity of restrictions of f to other octonionic lines follows from the hypothesis and the transitivity of the action of $Spin(9)$ on $\mathbb{O}\mathbb{P}^1$. Q.E.D.

3.1.15 Theorem. *Let $\Omega \subset \mathbb{O}^2$ be an open connected subset. Let $\{f_n\}$ be a sequence of octonionic plurisubharmonic functions in Ω which is uniformly bounded from above on every compact subset of Ω .*

(i) *Then either $f_n \rightarrow -\infty$ uniformly on every compact subset of Ω , or else there is a subsequence $\{f_{n_k}\}$ which converges in $L^1_{loc}(\Omega)$.*

(ii) *If $f_n \not\equiv -\infty$ for all n , and $\{f_n\}$ converges in the sense of generalized functions to $F \in C^{-\infty}(\Omega)$, then F is defined by an octonionic plurisubharmonic function $f \not\equiv -\infty$ and $f_n \rightarrow f$ in $L^1_{loc}(\Omega)$.*

Proof. Both statements follow immediately from the corresponding statements on subharmonic functions (Lemma 3.1.13) using Propositions 3.1.5, 3.1.6, and 3.1.14. Q.E.D.

3.1.16 Proposition. *Let $f: \Omega \rightarrow \mathbb{R} \cup \{-\infty\}$ be a function such that $f \not\equiv -\infty$. Then f is octonionic plurisubharmonic if and only if it satisfies the following conditions:*

- (i) *f is upper semi-continuous;*
- (ii) *f is locally integrable;*
- (iii) *$\left(\frac{\partial^2 f}{\partial q_i \partial \bar{q}_j}\right) \geq 0$ in the sense of generalized functions.*

Proof. Actually it remains to show that if a function $f \not\equiv -\infty$ is upper semi-continuous and locally integrable then f is octonionic plurisubharmonic if and only if $\left(\frac{\partial^2 f}{\partial q_i \partial \bar{q}_j}\right) \geq 0$ in the sense of generalized functions.

Let us fix a sequence $\{\delta_n\}$ approximating the δ -function at 0 as above. Then $f * \delta_n$ is infinitely smooth and converges to f weakly in sense of measures since $f \in L^1_{loc}(\Omega)$. Let us assume first that f is octonionic plurisubharmonic. Then $f * \delta_n$ is octonionic plurisubharmonic by Proposition 3.1.12(i). Hence by Proposition 3.1.8 $\left(\frac{\partial^2 (f * \delta_n)}{\partial q_i \partial \bar{q}_j}\right) \geq 0$ pointwise. But obviously

$$\frac{\partial^2 (f * \delta_n)}{\partial q_i \partial \bar{q}_j} = \frac{\partial^2 f}{\partial q_i \partial \bar{q}_j} * \delta_n.$$

Hence $\left(\frac{\partial^2(f*\delta_n)}{\partial q_i \partial \bar{q}_j}\right) \rightarrow \left(\frac{\partial^2 f}{\partial q_i \partial \bar{q}_j}\right)$ is the sense of generalized functions. Hence $\left(\frac{\partial^2 f}{\partial q_i \partial \bar{q}_j}\right) \geq 0$.

Conversely let us assume that $\left(\frac{\partial^2 f}{\partial q_i \partial \bar{q}_j}\right) \geq 0$. Then $\left(\frac{\partial^2(f*\delta_n)}{\partial q_i \partial \bar{q}_j}\right) = \left(\frac{\partial^2 f}{\partial q_i \partial \bar{q}_j}\right) * \delta_n \geq 0$. Since $f * \delta_n$ is infinitely smooth, it is octonionic plurisubharmonic by Proposition 3.1.8. Since $f * \delta_n \rightarrow f$ in the sense of generalized functions, the function f is octonionic plurisubharmonic by Theorem 3.1.15(ii). Q.E.D.

3.2 An analogue of the Aleksandrov and Chern-Levine-Nirenberg theorems.

In this section we will denote for brevity the octonionic Hessian $\left(\frac{\partial^2 u}{\partial q_i \partial \bar{q}_j}\right)$ by $\partial^2 u$.

Let us define a 3-linear functional τ on triples of infinitely smooth compactly supported \mathbb{R} -valued functions on \mathbb{O}^2 by

$$\tau(f_0, f_1, f_2) = \int_{\mathbb{O}^2} f_0 \det(\partial^2 f_1, \partial^2 f_2) dq$$

where dq is the standard Lebesgue measure. Later on we will use the following technical lemma.

3.2.1 Lemma. *τ is symmetric with respect to all 3 arguments f_0, f_1, f_2 .*

Proof. It is clear that τ is invariant with respect to f_1 and f_2 . It is enough to show that

$$\tau(f_0, f_1, f_2) = \tau(f_1, f_0, f_2). \quad (29)$$

It will be more convenient to prove (29) under slightly more general assumptions: we will assume that $f_0, f_2 \in C_0^\infty(\mathbb{O}^2)$, and $f_1 \in C^{-\infty}(\mathbb{O}^2)$. First let us prove (29) for $f_1 = \delta_{\{q_1=0\}}$, i.e. f_1 is the δ -function of the octonionic line $\{(0, x) | x \in \mathbb{O}\}$. We have

$$\begin{aligned} \partial^2 f_1 &= \begin{bmatrix} \Delta_1 \delta_{\{q_1=0\}} & 0 \\ 0 & 0 \end{bmatrix}, \\ \det(\partial^2 f_1, \partial^2 f_2) &= \frac{1}{2} \Delta_2 f_2 \cdot \Delta_1 \delta_{\{q_1=0\}} \end{aligned}$$

where Δ_i denotes the usual Laplacian with respect to the i -th octonionic variable.

Then

$$\tau(f_0, f_1, f_2) = \frac{1}{2} \int_{\mathbb{O}^2} f_0 \cdot \Delta_2 f_2 \cdot \Delta_1 \delta_{\{q_1=0\}} = \frac{1}{2} \int_{\{q_1=0\}} \Delta_1(f_0 \cdot \Delta_2 f_2) dq_2. \quad (30)$$

On the other hand

$$\begin{aligned} \tau(f_1, f_0, f_2) &= \int_{\mathbb{O}^2} f_1 \cdot \det(\partial^2 f_0, \partial^2 f_2) dq = \int_{\{q_1=0\}} \det(\partial^2 f_0, \partial^2 f_2) dq_2 = \\ &= \int_{\{q_1=0\}} \det \left(\begin{bmatrix} \Delta_1 f_0 & \frac{\partial^2 f_0}{\partial q_1 \partial \bar{q}_2} \\ \frac{\partial^2 f_0}{\partial q_2 \partial \bar{q}_1} & \Delta_2 f_0 \end{bmatrix}, \begin{bmatrix} \Delta_1 f_2 & \frac{\partial^2 f_2}{\partial q_1 \partial \bar{q}_2} \\ \frac{\partial^2 f_2}{\partial q_2 \partial \bar{q}_1} & \Delta_2 f_2 \end{bmatrix} \right) dq_2 = \\ &= \frac{1}{2} \int_{\{q_1=0\}} \left(\Delta_1 f_0 \cdot \Delta_2 f_2 + \Delta_1 f_2 \cdot \Delta_2 f_0 - 2 \operatorname{Re} \left(\frac{\partial^2 f_0}{\partial q_1 \partial \bar{q}_2} \cdot \frac{\partial^2 f_2}{\partial q_2 \partial \bar{q}_1} \right) \right) dq_2. \end{aligned}$$

After integration by parts in the second summand we obtain

$$\tau(f_1, f_0, f_2) = \frac{1}{2} \int_{\{q_1=0\}} \left(\Delta_1 f_0 \cdot \Delta_2 f_2 + f_0 \cdot \Delta_1 \Delta_2 f_2 - 2 \operatorname{Re} \left(\frac{\partial^2 f_2}{\partial q_2 \partial \bar{q}_1} \cdot \frac{\partial^2 f_0}{\partial q_1 \partial \bar{q}_2} \right) \right) dq_2 \quad (31)$$

Let us integrate the third summand in the last expression:

$$\begin{aligned} & \int_{\{q_1=0\}} \operatorname{Re} \left(\frac{\partial^2 f_2}{\partial q_2 \partial \bar{q}_1} \cdot \frac{\partial^2 f_0}{\partial q_1 \partial \bar{q}_2} \right) dq_2 = \\ & \int_{\{q_1=0\}} \operatorname{Re} \left(\left(\sum_{t=0}^7 \frac{\partial^2 f_2}{\partial x_2^t \partial \bar{q}_1} \bar{e}_t \right) \cdot \left(\sum_{s=0}^7 e_s \frac{\partial^2 f_0}{\partial x_2^s \partial q_1} \right) \right) dq_2 \quad \text{by parts} \\ & - \sum_{s,t=0}^7 \int_{\{q_1=0\}} \operatorname{Re} \left(\left(\frac{\partial^2}{\partial x_2^s \partial x_2^t} \left(\frac{\partial f_2}{\partial \bar{q}_1} \right) \bar{e}_t \right) \cdot \left(e_s \frac{\partial f_0}{\partial q_1} \right) \right) dq_2 \quad \text{Lemma 1.1.1(i)} \\ & - \sum_{s,t=0}^7 \int_{\{q_1=0\}} \operatorname{Re} \left(\left(\left(\frac{\partial^2}{\partial x_2^s \partial x_2^t} \left(\frac{\partial f_2}{\partial \bar{q}_1} \right) \bar{e}_t \right) e_s \right) \cdot \frac{\partial f_0}{\partial q_1} \right) dq_2 \quad \text{Lemma 1.1.1(iii)} \\ & - \sum_{s=0}^7 \int_{\{q_1=0\}} \operatorname{Re} \left(\left(\frac{\partial^2}{(\partial x_2^s)^2} \left(\frac{\partial f_2}{\partial \bar{q}_1} \right) \right) \cdot \frac{\partial f_0}{\partial q_1} \right) dq_2 - \\ & \sum_{0 \leq s < t \leq 7} \int_{\{q_1=0\}} \operatorname{Re} \left(\left(\left(\frac{\partial^2}{\partial x_2^s \partial x_2^t} \left(\frac{\partial f_2}{\partial \bar{q}_1} \right) \right) (\bar{e}_t e_s + \bar{e}_s e_t) \right) \cdot \frac{\partial f_0}{\partial q_1} \right) dq_2 = \\ & - \int_{\{q_1=0\}} \operatorname{Re} \left(\frac{\partial(\Delta_2 f_2)}{\partial \bar{q}_1} \cdot \frac{\partial f_0}{\partial q_1} \right) dq_2. \end{aligned}$$

Substituting the last expression back to (31) we obtain

$$\begin{aligned} \tau(f_1, f_0, f_2) &= \frac{1}{2} \int_{\{q_1=0\}} \left(\Delta_1 f_0 \cdot \Delta_2 f_2 + f_0 \cdot \Delta_1 \Delta_2 f_2 + 2 \operatorname{Re} \left(\frac{\partial(\Delta_2 f_2)}{\partial \bar{q}_1} \cdot \frac{\partial f_0}{\partial q_1} \right) \right) dq_2 = \\ & \frac{1}{2} \int_{\{q_1=0\}} \frac{\partial^2}{\partial q_1 \partial \bar{q}_1} (f_0 \cdot \Delta_2 f_2) dq_2 = \frac{1}{2} \int_{\{q_1=0\}} \Delta_1 (f_0 \cdot \Delta_2 f_2) dq_2 \quad \text{by (30)} \tau(f_0, f_1, f_2). \end{aligned}$$

Thus the equality (30) is proved for $f_1 = \delta_{\{q_1=0\}}$. Next using Proposition 1.5.1 and the fact that the group $\mathbb{O}^2 \rtimes Spin(9)$ acts transitively on \mathcal{AOP}^1 we conclude that the equality (29) for $f_1 = \delta_L$ for any $L \in \mathcal{AOP}^1$. But by Corollary 2.1.4 linear combinations of such δ -functions are dense in $C^{-\infty}(\mathbb{O}^2)$, hence the equality (29) is proved for any generalized function f_1 . Q.E.D.

For any matrix-valued function $F: \Omega \rightarrow \mathcal{H}_2(\mathbb{O})$ let us denote by $\|F\|_{L^1(\Omega)}$ its L^1 -norm, i.e. the sum of the L^1 -norms of the elements of this matrix. It is easy to see that if F takes values in non-negative definite matrices, then

$$\|F\|_{L^1(\Omega)} \leq 2 \|\det(F, I_2)\|_{L^1(\Omega)} = 2 \int_{\Omega} \det(F, I_2) \quad (32)$$

where $I_2 \in \mathcal{H}_2(\mathbb{O})$ is the identity matrix.

3.2.2 Lemma. *For any compact subset $K \subset \Omega$ and any its compact neighborhood $K' \subset \Omega$ there exists a constant C such that for any $f \in P(\Omega) \cap C^2(\Omega)$ one has*

$$\|(\partial^2 f)\|_{L^1(K)} \leq C\|f\|_{L^\infty(K')}; \quad (33)$$

$$\|\det(\partial^2 f)\|_{L^1(K)} \leq C\|f\|_{L^\infty(K')}^2. \quad (34)$$

Proof. Let us fix a non-negative function $\gamma \in C_0^\infty(\Omega)$ which is equal to 1 in a neighborhood of K and vanishes on a neighborhood of the closure $\overline{\Omega \setminus K'}$. Then using (32) one has

$$\begin{aligned} \|(\partial^2 f)\|_{L^1(K)} &\leq 2 \int_{\Omega} \gamma \det(\partial^2 f, I_2) = \\ &= \int_{\Omega} \gamma \cdot \Delta f = \int_{\Omega} f \Delta \gamma \leq C\|f\|_{L^\infty(K')}. \end{aligned}$$

Thus the inequality (33) is proved. Let us prove (34). We have

$$\begin{aligned} \|\det(\partial^2 f)\|_{L^1(K)} &\leq \int_{\Omega} \gamma \det(\partial^2 f) \stackrel{\text{Lemma 3.2.1}}{=} \int_{\Omega} f \det(\partial^2 \gamma, \partial^2 f) \leq \\ &\leq C'\|f\|_{L^\infty(K')} \cdot \|(\partial^2 f)\|_{L^1(\text{supp } \gamma)} \leq C''\|f\|_{L^\infty(K')}^2 \end{aligned}$$

where the last inequality follows from (33). Lemma is proved. Q.E.D.

3.2.3 Corollary. *For any compact subset $K \subset \Omega$ and any its compact neighborhood $K' \subset \Omega$ there exists a constant C such that for any $f, g \in P(\Omega) \cap C^2(\Omega)$ and any $\phi \in C^2(\Omega)$ with $\text{supp } \phi \subset K$ one has*

$$\left| \int_{\Omega} \phi (\det(\partial^2 f) - \det(\partial^2 g)) \right| \leq C\|f - g\|_{L^\infty(K)} (\|f\|_{L^\infty(K')} + \|g\|_{L^\infty(K')}) \cdot \|\phi\|_{C^2(\Omega)}. \quad (35)$$

Proof. We have

$$\begin{aligned} &\left| \int_{\Omega} \phi (\det(\partial^2 f) - \det(\partial^2 g)) \right| \leq \\ &\left| \int_{\Omega} \phi \det(\partial^2(f - g), \partial^2 f) \right| + \left| \int_{\Omega} \phi \det(\partial^2 g, \partial^2(f - g)) \right| \stackrel{\text{Lemma 3.2.1}}{=} \\ &\left| \int_{\Omega} (f - g) \det(\partial^2 f, \partial^2 \phi) \right| + \left| \int_{\Omega} (f - g) \det(\partial^2 g, \partial^2 \phi) \right| \leq \\ &C\|f - g\|_{L^\infty(K)} \|\phi\|_{C^2(\Omega)} (\|(\partial^2 f)\|_{L^1(K)} + \|(\partial^2 g)\|_{L^1(K)}) \stackrel{\text{Lemma 3.2.2}}{\leq} \\ &C'\|f - g\|_{L^\infty(K)} \|\phi\|_{C^2(\Omega)} (\|f\|_{L^\infty(K')} + \|g\|_{L^\infty(K')}). \end{aligned}$$

Corollary is proved. Q.E.D.

3.2.4 Proposition. *Let $\Omega \subset \mathbb{O}^2$ be an open subset. Assume that a sequence $\{f_n\} \subset P(\Omega)$, $f_n \not\equiv -\infty$ for any n , converges in $L_{loc}^1(\Omega)$ to an octonionic plurisubharmonic function $f \not\equiv -\infty$. Then one has a weak convergence of $\mathcal{H}_2(\mathbb{O})$ -valued measures*

$$\left(\frac{\partial^2 f_n}{\partial q_i \partial \bar{q}_j} \right) \rightarrow \left(\frac{\partial^2 f}{\partial q_i \partial \bar{q}_j} \right).$$

Proof. By Lemma 3.2.2 the measures $\{(\frac{\partial^2 f_n}{\partial q_i \partial \bar{q}_j})\}$ are uniformly locally bounded in Ω . Hence choosing a subsequence if necessary we may assume that this sequence of measures converges weakly to an $\mathcal{H}_2(\mathbb{O})$ -valued measure $(\nu_{i\bar{j}})$. We have to prove that $\nu_{i\bar{j}} = \frac{\partial^2 f}{\partial q_i \partial \bar{q}_j}$. To see it let us fix an arbitrary function $\phi \in C_0^\infty(\Omega)$. Then

$$\int_{\Omega} \frac{\partial^2 f_n}{\partial q_i \partial \bar{q}_j} \cdot \phi dq = \int_{\Omega} f_n \cdot \frac{\partial^2 \phi}{\partial q_i \partial \bar{q}_j} dq \rightarrow \int_{\Omega} f \cdot \frac{\partial^2 \phi}{\partial q_i \partial \bar{q}_j} dq = \int_{\Omega} \frac{\partial^2 f}{\partial q_i \partial \bar{q}_j} \phi \cdot dq,$$

where the first and the last equalities are obtained by integration by parts. The result follows. Q.E.D.

3.2.5 Proposition. *Let $f \in P(\Omega) \cap C(\Omega)$. There exists a unique measure on Ω denoted by $\det(\partial^2 f)$ satisfying the following property: for any sequence $\{f_n\} \subset P(\Omega) \cap C^2(\Omega)$ such that $f_n \rightarrow f$ uniformly on compact subsets of Ω one has $\det(\partial^2 f_n) \rightarrow \det(\partial^2 f)$ weakly in sense of measures. This measure $\det(\partial^2 f)$ is non-negative and has the obvious interpretation when $f \in C^2(\Omega)$.*

Proof. By Lemma 3.2.2 for any compact subset $K \subset \Omega$ and any its compact neighborhood K' there exists a constant C such that for any n

$$\|\det(\partial^2 f_n)\|_{L^1(K)} \leq C \|f_n\|_{L^\infty(K')}^2.$$

Hence the sequence of measures $\det(\partial^2 f_n)|_K$ on K has a compact closure in the weak topology. It remains to show that this sequence has at most one limiting point. By Corollary 3.2.3 we have for any $\phi \in C_0^\infty(\Omega)$ with $\text{supp } \phi \subset K$

$$\left| \int_{\Omega} \phi (\det(\partial^2 f_n) - \det(\partial^2 f_m)) \right| \leq C \|\phi\|_{C^2(\Omega)} \|f_n - f_m\|_{L^\infty(K)} (\|f_n\|_{L^\infty(K')} + \|f_m\|_{L^\infty(K')}). \quad (36)$$

Obviously the last expression tends to 0 as $m, n \rightarrow \infty$. This implies also that the limiting measure is unique and it is independent of the approximating sequence $\{f_n\}$. Let us denote it temporarily by μ .

Obviously the limiting measure μ is non-negative. It remains to check that if $f \in C^2(\Omega)$ then $\mu = \det(\partial^2 f)$. We have shown that μ is independent of an approximating sequence. Taking the constant sequence equal to f we conclude the result. Q.E.D.

The next lemma generalizes Lemma 3.2.2 and Corollary 3.2.3 to the class of all continuous plurisubharmonic functions.

3.2.6 Lemma. *For any compact subset $K \subset \Omega$ and any its compact neighborhood $K' \subset \Omega$ there exists a constant C such that*

(i) *for any $f \in P(\Omega) \cap C(\Omega)$ one has*

$$\|(\partial^2 f)\|_{L^1(K)} \leq C \|f\|_{L^\infty(K')}, \quad (37)$$

$$\|\det(\partial^2 f)\|_{L^1(K)} \leq C \|f\|_{L^\infty(K')}^2; \quad (38)$$

(ii) *for any $f, g \in P(\Omega) \cap C(\Omega)$ and any $\phi \in C^2(\Omega)$ with $\text{supp } \phi \subset K$ one has*

$$\left| \int_{\Omega} \phi (\det(\partial^2 f) - \det(\partial^2 g)) \right| \leq C \|f - g\|_{L^\infty(K)} (\|f\|_{L^\infty(K')} + \|g\|_{L^\infty(K')}) \cdot \|\phi\|_{C^2(\Omega)}. \quad (39)$$

Proof. This lemma follows immediately from Lemma 3.2.2 and Corollary 3.2.3 using Proposition 3.2.5 and the approximation of f by smooth plurisubharmonic functions as in Proposition 3.1.12. Q.E.D.

3.2.7 Theorem. *Let $\Omega \subset \mathbb{O}^2$ be an open subset. Let a sequence $\{f_n\} \subset P(\Omega) \cap C(\Omega)$ converges uniformly on compact subsets of Ω to a function f . Then $f \in P(\Omega) \cap C(\Omega)$ and $\det(\partial^2 f_n) \rightarrow \det(\partial^2 f)$ weakly in sense of measures.*

Proof. This theorem follows from Lemma 3.2.6 exactly in the same way as Proposition 3.2.5 followed from Lemma 3.2.2 and Corollary 3.2.3. Q.E.D.

From Proposition 3.2.5 and Theorem 3.2.7 one can easily deduce the following 'mixed' version of these results generalizing both of them. This version uses the notion of mixed determinant introduced in Section 1.2.

3.2.8 Theorem. *Let $\Omega \subset \mathbb{O}^2$ be an open subset. For any $f, g \in P(\Omega) \cap C(\Omega)$ there exists a non-negative measure denoted by $\det(\partial^2 f, \partial^2 g)$. It satisfies and is uniquely characterized by the following two properties:*

- (i) *if $f, g \in C^2(\Omega)$ then the meaning is obvious;*
- (ii) *if one has two sequences $\{f_n\}, \{g_n\} \subset P(\Omega) \cap C(\Omega)$ such that $f_n \rightarrow f, g_n \rightarrow g$ uniformly on compact subsets of Ω then $\det(\partial^2 f_n, \partial^2 g_n) \rightarrow \det(\partial^2 f, \partial^2 g)$ weakly in the sense of measures.*

Note that non-negativity of measures follows from Lemma 1.2.6.

3.3 A Błocki type theorem.

3.3.1 Theorem. *For any $u, v \in P(\Omega) \cap C(\Omega)$ one has*

$$\det(\partial^2(\max\{u, v\})) = \det(\partial^2(\max\{u, v\}), \partial^2 u + \partial^2 v) - \det(\partial^2 u, \partial^2 v). \quad (40)$$

Proof. The argument is very close to the original Błocki's argument [21]. By continuity of both sides in (40) we may assume that u, v are smooth. Let $\chi : \mathbb{R} \rightarrow [0, \infty)$ be a smooth function such that $\chi(x) = 0$ if $x \leq -1$, $\chi(x) = x$ if $x \geq 1$, and $0 \leq \chi' \leq 1$, $\chi'' \geq 0$ everywhere. Define

$$\begin{aligned} \psi_j &:= v + \frac{1}{j} \chi(j(u - v)), \\ \alpha &:= u - v, \\ w &:= \max\{u, v\}. \end{aligned}$$

It is easy to see that $\psi_j \downarrow w$ uniformly on compact subsets and monotonically as $j \rightarrow \infty$.

3.3.2 Lemma.

$$\left(\frac{\chi(j\alpha)}{j} \right)_{p\bar{q}} = \chi'(j\alpha) \cdot \alpha_{p\bar{q}} + j\chi''(j\alpha) \alpha_{\bar{q}} \alpha_p.$$

Proof. Since α is a real valued function we have

$$\begin{aligned} \left(\frac{\chi(j\alpha)}{j} \right)_{p\bar{q}} &= \frac{1}{j} \sum_{l,m=0}^7 e_l(\chi(j\alpha))_{x_q^l x_p^m} \bar{e}_m = \\ \sum_{l,m=0}^7 e_l \left(\chi'(j\alpha) \cdot \alpha_{x_q^l} \right)_{x_p^m} \bar{e}_m &= \sum_{l,m=0}^7 e_l \left(j\chi''(j\alpha) \cdot \alpha_{x_q^l} \alpha_{x_p^m} + \chi'(j\alpha) \alpha_{x_q^l x_p^m} \right) \bar{e}_m = \\ \chi'(j\alpha) \alpha_{p\bar{q}} + j\chi''(j\alpha) \alpha_{\bar{q}} \alpha_p. \end{aligned}$$

Q.E.D.

Thus from Lemma 3.3.2 we obtain

$$\begin{aligned} (\psi_j)_{p\bar{q}} &= v_{p\bar{q}} + \chi'(j\alpha)(u - v)_{p\bar{q}} + j\chi''(j\alpha) \alpha_{\bar{q}} \alpha_p = \\ \chi'(j\alpha) u_{p\bar{q}} + (1 - \chi'(j\alpha)) v_{p\bar{q}} + j\chi''(j\alpha) \alpha_{\bar{q}} \alpha_p. \end{aligned}$$

The matrix $(\alpha_{\bar{q}} \alpha_p)$ is non-negative definite. Then, since $0 \leq \chi' \leq 1$ and $\chi'' \geq 0$, it follows that ψ_j is plurisubharmonic. From the definition of ψ_j we have

$$\det(\partial^2 \psi_j) = \det(\partial^2 v) + 2 \det \left(\partial^2 v, \partial^2 \left(\frac{\chi(j\alpha)}{j} \right) \right) + \det \left(\partial^2 \left(\frac{\chi(j\alpha)}{j} \right) \right). \quad (41)$$

We have weak convergence

$$\det(\partial^2 \psi_j) \rightarrow \det(\partial^2 w), \quad (42)$$

$$\det \left(\partial^2 v, \partial^2 \left(\frac{\chi(j\alpha)}{j} \right) \right) \rightarrow \det(\partial^2(w - v), \partial^2 v). \quad (43)$$

Let us study the last term in (41), namely $\det \left(\partial^2 \left(\frac{\chi(j\alpha)}{j} \right) \right)$. From Lemma 3.3.2 one gets

$$\det \left(\partial^2 \left(\frac{\chi(j\alpha)}{j} \right) \right) = \quad (44)$$

$$\chi'(j\alpha)^2 \det(\alpha_{p\bar{q}}) + 2j\chi'(j\alpha)\chi''(j\alpha) \det((\alpha_{\bar{q}} \alpha_p), (\alpha_{p\bar{q}})) + (j\chi''(j\alpha))^2 \det(\alpha_{\bar{q}} \alpha_p) = \quad (45)$$

$$\chi'(j\alpha)^2 \det(\alpha_{p\bar{q}}) + 2j\chi'(j\alpha)\chi''(j\alpha) \det((\alpha_{\bar{q}} \alpha_p), (\alpha_{p\bar{q}})) \quad (46)$$

since the last summand in (45) vanishes. Let $\gamma : \mathbb{R} \rightarrow \mathbb{R}$ be such that $\gamma' = (\chi')^2$. Then we have

3.3.3 Lemma.

$$\det \left(\partial^2 \left(\frac{\chi(j\alpha)}{j} \right) \right) = \det \left(\partial^2 \left(\frac{\gamma(j\alpha)}{j} \right), \partial^2 \alpha \right).$$

Let us postpone the proof of Lemma 3.3.3, and finish the proof of Theorem 3.3.1. One can choose γ so that $\gamma(-1) = 0$. Then $\frac{\gamma(jx)}{j} \downarrow \max\{0, x\}$ uniformly on compact subsets and monotonically as $j \rightarrow \infty$. Hence

$$\det \left(\partial^2 \left(\frac{\chi(j\alpha)}{j} \right) \right) \rightarrow \det(\partial^2(w - v), \partial^2 \alpha) \text{ weakly.}$$

This and (41), (42), (43) imply

$$\begin{aligned}\det(\partial^2 w) &= \det(\partial^2 v) + 2\det(\partial^2(w-v), \partial^2 v) + \det(\partial^2(w-v), \partial^2(u-v)) \\ &= \det(\partial^2 w, (\partial^2 u + \partial^2 v)) - \det(\partial^2 u, \partial^2 v).\end{aligned}$$

This implies Theorem 3.3.1. It remains to prove Lemma 3.3.3.

Proof of Lemma 3.3.3. We have

$$\begin{aligned}\left(\frac{\gamma(j\alpha)}{j}\right)_{p\bar{q}} &= \gamma'(j\alpha)\alpha_{p\bar{q}} + \alpha_{\bar{q}} \cdot (\gamma'(j\alpha))_p \\ &= (\chi'(j\alpha))^2 \alpha_{p\bar{q}} + 2j\chi'(j\alpha)\chi''(j\alpha)\alpha_{\bar{q}} \cdot \alpha_p.\end{aligned}$$

This and the equality (46)=(44) imply Lemma 3.3.3. Q.E.D.

3.3.4 Corollary. *For any $u, v \in P(\Omega) \cap C(\Omega)$ such that $\min\{u, v\} \in P(\Omega)$ one has*

$$\det(\partial^2(\min\{u, v\})) = \det(\partial^2 u) + \det(\partial^2 v) - \det(\partial^2(\max\{u, v\})).$$

Proof. Observe that $\min\{u, v\} = u + v - \max\{u, v\}$. Denote for brevity $U := \partial^2 u, V := \partial^2 v, W := \partial^2(\max\{u, v\})$. Then we get

$$\begin{aligned}\det(\partial^2(\min\{u, v\})) &= \det(U + V - W) = \\ &= (\det U + \det V - \det W) + \\ &+ 2\det W + 2\det(U, V) - 2\det(U, W) - 2\det(V, W) \stackrel{\text{Theorem 3.3.1}}{=} \\ &= \det U + \det V - \det W.\end{aligned}$$

Corollary is proved. Q.E.D.

4 Valuations on convex subsets of \mathbb{O}^2 .

4.1 "Obvious" examples of valuations on \mathbb{O}^2 .

We denote by $Val(\mathbb{O}^2)$ the space of translation invariant continuous valuations on convex compact subsets of \mathbb{O}^2 . We denote by $Val^{Spin(9)}(\mathbb{O}^2)$ the subspace of $Spin(9)$ -invariant valuations. Since the group $Spin(9)$ acts transitively on the unit sphere S^{15} , the space $Val^{Spin(9)}(\mathbb{O}^2)$ is finite dimensional by [4], Theorem 8.1. Since $-Id \in Spin(9)$ all valuations in $Val^{Spin(9)}(\mathbb{O}^2)$ are even, i.e. they take the same value on K and $-K$ for any convex compact set K .

For the moment we can neither classify valuations in $Val^{Spin(9)}(\mathbb{O}^2)$ nor even compute the dimension of this space. The goal of this section is to present few examples of such valuations which are natural from the point of view of convexity and integral geometry.

For a non-negative integer $i \leq 16$ let us denote by V_i the intrinsic volume which is a continuous valuation invariant under all isometries of the Euclidean space \mathbb{R}^{16} . By definition, for any $K \in \mathcal{K}(\mathbb{O}^2)$ the intrinsic volume $V_i(K)$ is equal (up to a normalizing constant) to the mixed volume $V(\underbrace{K, \dots, K}_i, \underbrace{D, \dots, D}_{16-i})$ where D is the unit Euclidean ball. We refer

to the book [45] for the details on the mixed and intrinsic volumes. Thus in particular $V_i \in Val^{Spin(9)}(\mathbb{O}^2)$. Recall that V_{16} is proportional to the Lebesgue measure, and V_0 is the Euler characteristic (which is equal to 1 on any convex compact set).

To construct more examples let us fix $i = 0, 1, \dots, 8$. Define

$$T_i(K) := \int_{E \in \mathbb{OP}^1} V_i(pr_E(K)) dE \quad (47)$$

where $pr_E: \mathbb{O}^2 \rightarrow E$ is the orthogonal projection, and dE is the probability $Spin(9)$ -invariant Haar measure on \mathbb{OP}^1 . It is easy to see that $T_i \in Val^{Spin(9)}(\mathbb{O}^2)$. Moreover T_i is i -homogeneous: a valuation ϕ is called i -homogeneous if $\phi(\lambda K) = \lambda^i \phi(K)$ for any $\lambda > 0$ and any set K .

Next fix $8 \leq j \leq 16$. Define

$$U_j(K) := \int_{F \in \mathcal{AO}\mathbb{P}^1} V_{j-8}(K \cap F) dF \quad (48)$$

where dF is $\mathbb{O}^2 \rtimes Spin(9)$ -invariant Haar measure on $\mathcal{AO}\mathbb{P}^1$ (we do not specify the normalization of this measure since it is irrelevant for the moment). Then $U_j \in Val^{Spin(9)}(\mathbb{O}^2)$. U_j is j -homogeneous.

4.1.1 Remark. The valuations T_i and U_{16-i} are Fourier transform of each other (up to a constant). For the notion of the Fourier transform on valuations we refer to [8] for the even case (where this transform is called the duality operator) and to [14] for the general case. Notice that the intrinsic volumes V_i and V_{16-i} are also Fourier transforms of each other (up to a constant).

4.1.2 Remark. $Val^{Spin(9)}(\mathbb{O}^2)$ has a natural product [9] making it a commutative associative graded algebra with unit (where the unit is the Euler characteristic). Thus taking polynomials in the above examples of valuations we can produce more examples of $Spin(9)$ -invariant valuations. Moreover one can take convolutions in sense of [20] of the above examples. However at present relations of these examples to the previous ones are not known (e.g. which of them are linearly independent? do they span $Val^{Spin(9)}(\mathbb{O}^2)$?).

4.1.3 Remark. There is still yet another general construction of valuations which is based on integration of $Spin(9)$ -invariant differential forms on the spherical cotangent bundle of \mathbb{O}^2 with respect to the normal cycle of a convex set. It was shown in [12], Theorem 5.2.1, that this construction gives all valuations from $Val^{Spin(9)}(\mathbb{O}^2)$. This construction of valuations uses results of J. Fu [25], [26]; more details in the context of valuations are given in [15]. Relations of this construction to other ones discussed in this article have not been studied.

4.2 New examples of valuations on \mathbb{O}^2 .

Recall that the space \mathbb{O}^2 is equipped with the standard Euclidean product

$$\langle (q_1, q_2), (z_1, z_2) \rangle = Re(q_1 \bar{z}_1 + q_2 \bar{z}_2).$$

For a convex compact set $K \in \mathcal{K}(\mathbb{O}^2)$ one defines its *supporting functional* $h_K: \mathbb{O}^2 \rightarrow \mathbb{R}$ by

$$h_K(x) := \sup_{y \in K} \langle x, y \rangle .$$

Then h_K is a convex 1-homogeneous function. In particular it is octonionic plurisubharmonic (Example 3.1.4).

4.2.1 Theorem. *Fix a continuous compactly supported function ψ on \mathbb{O}^2 . Then*

$$K \mapsto \int_{\mathbb{O}^2} \det \left(\frac{\partial^2 h_K}{\partial q_i \partial \bar{q}_j} \right) \cdot \psi dq$$

is a translation invariant continuous 2-homogeneous valuation on $\mathcal{K}(\mathbb{O}^2)$.

Proof. Translation invariance is obvious. Continuity follows from Theorem 3.2.7. To prove the valuation property let us observe first that if $K = K_1 \cup K_2$ with $K_1, K_2, K \in \mathcal{K}(\mathbb{O}^2)$ then

$$h_K = \max\{h_{K_1}, h_{K_2}\}, \quad h_{K_1 \cap K_2} = \min\{h_{K_1}, h_{K_2}\}.$$

Hence the result follows from Theorem 3.3.1. Q.E.D.

From Theorem 4.2.1 and Proposition 1.5.1 we immediately deduce the following corollary.

4.2.2 Corollary. *The correspondence*

$$K \mapsto \int_D \det \left(\frac{\partial^2 h_K}{\partial q_i \partial \bar{q}_j} \right) dq$$

where D is the unit ball in \mathbb{O}^2 , is a $Spin(9)$ -invariant translation invariant continuous 2-homogeneous valuation on $\mathcal{K}(\mathbb{O}^2)$.

4.2.3 Remark. It is not hard to see that the valuation from Corollary 4.2.2 is *not* invariant under the group $SO(16)$. In particular it is not proportional to the second intrinsic volume V_2 on \mathbb{R}^{16} .

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